Florian Cajori.

INDUCTIVE

PLANE GEOMETRY

WITH

NUMEROUS EXERCISES, THEOREMS, AND PROBLEMS
FOR ADVANCE WORK

BY

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PREFACE.

The inductive method of teaching geometry is recognized as the ideal one by all progressive teachers. Committing to memory the demonstration of another is acknowledged to be of little benefit to the pupil. Development of the reasoning powers comes only from their judicious exercise; and this is possible only in small degree from merely yielding assent to the logic of another.

In an experience of twenty years the author has found that fully three fourths of his pupils can demonstrate unaided, or at most with a suggestion or two, the majority of the theorems, the demonstrations of which are given in most text-books for the pupil to read and memorize. Consequently he has endeavored to be consistent, and offer aid in the way of suggestions only where the pupil needs it. In most text-books the proof of the following easy theorem is given for the pupil to memorize; viz.: "If two parallel lines are cut by a third straight line the exterior interior angles are equal." On the very same page are given others, far more difficult, for the pupil to prove unaided. The same condition of things is found on many subsequent pages. The author's experience has been that nine tenths of his pupils demonstrate the above theorem correctly unaided, and that the other tenth need only a single suggestion. The same is true of a large number of the theorems and construction problems usually given in full.

In this revised edition, the arrangement of the sequence of theorems has been radically changed in many places, noticeably in placing the subject of triangles as near the beginning as possible. This is done because of the great advantage in the

use of equality of triangles in subsequent demonstrations. This aids in reducing to a minimum the method of superposition which usually confuses the beginner.

The author has endeavored to remedy the defects of the old edition, and in this has been materially aided by various suggestions from numerous teachers who are in sympathy with the plan, and to whom he takes this occasion to express his thanks. Thanks are also due the publishers for the excellence of the mechanical work, as well as for many suggestions as to arrangement.

Finally, one of the most important features, in the estimation of the author, and one which he hopes will commend itself to teachers, is the printing the *essential theorems* in different type from the rest, so that a glance at the page will disclose them; *e.g.*:

If two sides and included angle of one triangle are equal respectively to two sides and included angle of another, the two triangles are equal.

Many of the others are helpful in proving subsequent theorems, while all are useful as exercises.

G. I. H.

MANCHESTER, N.H.,
June, 1902.

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PLANE GEOMETRY.

INTRODUCTION.

1. Space is indefinite extension in every direction.

2. A Material Substance is anything, large or small, solid, liquid, or aeriform, visible or invisible, that occupies space.

It therefore follows that material substances have *limited extension* in every direction.

3. For purposes of measurement, extension in *three* directions only are considered, called, respectively, *length*, *breadth*, and *thickness;* they are also called, collectively, *dimensions*.

4. Magnitude, in general, means size, and is applied to anything of which greater or less can be predicated, as time, weight, distance, etc.

A Geometrical Magnitude is that which has one or more of the three dimensions, as lines, angles, etc.

5. A Geometrical Point has position merely; *i.e.* it has no magnitude.

The dots made by pencil and crayon are *called* points, but they are really small substances used to indicate to the eye the location of the geometrical point.

6. A Geometrical Line has only one dimension; *i.e.* length.

The lines made by pencil and crayon are substances, and may be called *physical lines*, which serve to show the position of the geometrical lines.

7. A Straight Line is one that lies evenly between its extreme points.

This is the definition as given by Euclid. The majority of modern geometers, however, have substituted the following as stated by Newcomb, viz.:

"*A straight line* is one which has the same direction throughout its whole length."

Each is designed to express the idea of straightness, and not to convey it; for it is assumed that the idea already exists in the pupil's mind prior to the beginning of this study.

A straight line may also be defined as an *undeviating line.*

8. A Curved Line, or simply *curve,* is one no part of which is straight.

9. Material substances have one or more faces which separate them from the rest of space. These faces are called *surfaces,* and have, obviously, only two dimensions; *i.e. length* and *breadth.*

The surface considered apart from the substance is called a *geometrical surface.*

10. A Plane is a *geometrical surface* such that, if any two points in it be selected at random, the straight line joining them will lie wholly in that surface.

11. A Curved Surface is a geometrical surface no portion of which is a plane.

12. A physical solid is the material composing it, and which we perceive through the medium of the senses; while the *geometrical solid* is the *space,* simply, which the physical solid occupies.

13. A Geometrical Figure is the term applied to combinations of points, lines, and surfaces, when reference is had to their form or outline simply.

14. **A Plane Figure** is one whose points and lines all lie in the same plane.

15. "**A Plane Rectilineal Angle** is the inclination of two straight lines to one another, which meet together, but are not in the same straight line."—EUCLID.

"**An Angle** is a figure formed by two straight lines drawn from the same point."—CHAUVENET.

"When two straight lines meet together, their mutual inclination, or degree of opening, is called an *angle*."—LOOMIS.

16. The lines that form an angle are called the *sides* of an angle, and the point from which they are drawn is called the *vertex* of the angle.

17. When two plane angles have the same vertex and a common side, *neither angle being a part of the other*, they are said to be *adjacent* angles.

18. When two angles have the same vertex and the sides of one are the extensions of the sides of the other, they are called *vertical* angles.

19. An angle is named by a letter or number placed at its vertex. If, however, there are two or more angles with the same vertex, other letters are placed at the extremities of their sides, and the three letters are used to name the angle, *the letter at the vertex always coming between the other two.*

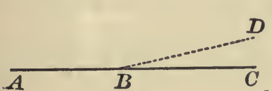


FIG. I.

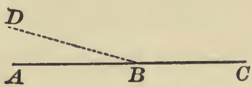


FIG. II.

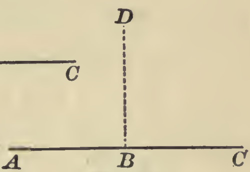


FIG. III.

20. Let us consider the point B, in the straight line AC, a pivot, and BD another starting from the position BC, and turning about B, keeping always in the same plane. It is

evident that, as soon as it has started, it forms *two angles* with the line AC, of which DBC, Fig. I, is the *smaller*. If it continues to revolve, however, it will finally reach a position as DB, Fig. II, in which the angle DBC is the *larger*. Hence, in passing from the first position to the second, it must have reached a position, Fig. III, where the two angles DBC and DBA were equal; hence,

21. When one straight line meets another so as to form *equal adjacent angles,* each of the angles is called a *right angle,* and the lines are said to be *perpendicular* to each other.

22. To say that an angle is a right angle therefore expresses *the same fact* as to say that two lines are perpendicular to each other. The equivalence of these two statements should be clearly noted and emphasized. It should also be borne in mind that the word *perpendicular* expresses a mutual relation between two lines, and therefore a right angle may be defined as *an angle whose sides are perpendicular to each other.*

23. It is evident that the sum of the angles formed by any one position of the line BD is equal to the sum of the angles formed by any other position; for what is taken from one angle by the revolution of the line BD is added to the other; hence,

24. When one straight line meets another so as to form two angles, the sum of these two angles equals two right angles.

25. It is also evident that there can only be *one* such position of the line BD; hence,

From a point *in* a straight line *only one perpendicular* to that line can be drawn, or if more be drawn they will all coincide.

26. When one straight line meets another so as to form *unequal adjacent angles,* the lines are said to be oblique to each other.

It should be clearly borne in mind by the pupil that the word *oblique* also in this connection expresses a *mutual* relation between the two lines.

27. An angle that is less than a right angle is called an *acute* angle.

28. An angle that is greater than one right angle and less than two is called an *obtuse* angle.

29. Both acute and obtuse angles are designated as *oblique* angles as contrasted with right angles.

30. In some discussions it is convenient to use other kinds of angles; *i.e.:*

A straight angle is defined as an angle whose sides extend in *exactly opposite directions from the vertex.*

31. Again, it is sometimes convenient to regard an angle as formed by a line revolving about one end as a pivot.

A complete revolution of such a line is called a *perigon*, or round angle.

A reflex angle is one that is larger than a straight angle and smaller than a perigon.

32. When the sum of two angles is equal to *two right angles,* they are said to be *supplemental; i.e.* each is the supplement of the other.

33. When the sum of two angles is equal to one right angle, they are said to be *complemental; i.e.* each is the complement of the other.

34. When the sum of two angles equals a perigon, they are said to be *explemental; i.e.* each is the explement of the other.

35. It is evident from 23 and 24 that when one straight line meets the other so as to form two angles, these angles are supplemental.

36. Two straight lines are said to be *parallel* when, lying in the same plane, and extended indefinitely both ways, they never meet each other.

37. As we have before conceived a line (20) to move, so we may conceive one geometrical magnitude to be applied to another for the purpose of comparison. If they coincide, point for point, they are said to be *congruent.*

38. It is evident that two geometrical magnitudes that are congruent are equal in all respects; hence,

39. If two angles can be so placed that their vertices coincide in position and their sides in direction, two and two, the angles are equal.

40. If two angles are equal and one be applied to the other so that their vertices and one pair of sides coincide, then the other pair of sides will also coincide.

41. Thus, if one right angle be applied to another so that their vertices coincide and likewise one pair of sides, then, since by Sec. 25 there can be only one perpendicular drawn to a line from a given point in it, the other two sides will also coincide; hence,

42. *All right angles are equal.*

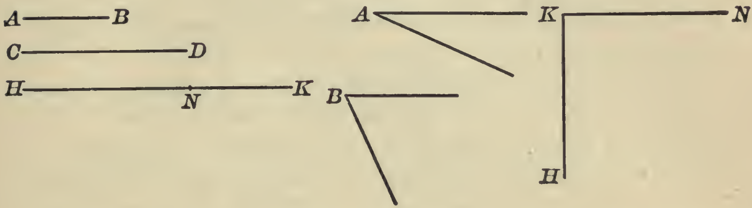
43. The word *coincide*, when used to express the relation of two lines, does not mean that they necessarily have the same extremities; while the word *congruent* does mean that.

44. **Geometrical Magnitudes** are geometrical lines, angles, surfaces, and solids.

45. We shall have occasion to express the addition and subtraction of geometrical magnitudes as well as the multiplication and division of these magnitudes by numbers.

46. For example, the sum of the two lines AB and CD is obtained by conceiving them to be placed so as to form one

continuous straight line as HK. Similarly, the difference of two lines is obtained by cutting off from the larger a line equal to the smaller. Similarly, HKN represents the sum of the two angles A and B. To multiply a line by a number is to add it to itself the required number of times.



To divide a line by a number is to conceive the line to be divided into the required number of equal parts.

The same is true of other geometrical magnitudes.

47. **An Axiom** is a truth that needs no argument; *i.e.* the mere statement of it makes it apparent, *e.g.*:

48. GENERAL AXIOMS.

i. The whole of anything is greater than any one of its parts.

ii. The whole of anything is equal to the sum of all its parts.

iii. Magnitudes which are equal to the same or equal magnitudes are equal to each other.

iv. Magnitudes which are halves of the same or equal magnitudes are equal to each other.

v. Magnitudes which are doubles of the same or equal magnitudes are equal to each other.

vi. If the same or equal quantities be added to equal quantities, the sums are equal.

vii. If the same or equal quantities be subtracted from equal quantities, the remaining quantities are equal.

viii. If equal quantities be multiplied by the same or equal quantities, the products are equal.

ix. If equal quantities be divided by the same or equal quantities, the quotients are equal.

x. If equal quantities be added to unequal quantities, the results will be unequal in the same order.

xi. If equal quantities be subtracted from unequal quantities, the results will be unequal in the same order.

xii. If unequal quantities be subtracted from equal quantities, the results will be unequal in the reverse order.

xiii. If unequal quantities be multiplied by the same or equal quantities, the products will be unequal in the same order.

xiv. If unequal quantities be divided by the same or equal quantities, the quotients will be unequal in the same order.

xv. Of two unequal quantities, a half of the larger is greater than a half of the smaller.

xvi. If unequals be added to unequals, the lesser to the lesser, and the greater to the greater, the sums will be unequal in the same order.

xvii. If two magnitudes are equal, either may be substituted for the other.

xviii. In an inequality a larger quantity may be substituted for the larger and a smaller for a smaller.

49. PARTICULAR AXIOMS.

xix. Between two points only one straight line can be drawn; or if others be drawn, they will all coincide.

xx. A straight line is the shortest line connecting two points.

xxi. Conversely, the shortest line between two points is a straight line.

xxii. If two straight lines have two points in common, they will coincide however far extended.

xxiii. Two straight lines can intersect in only one point.

xxiv. Through a given point (as P) only one line (as AB) can be drawn parallel to another line (as CD); or if others be drawn, they will all coincide.

xxv. If a line makes an angle with one of two parallel lines, it will intersect the other if sufficiently extended. (See above diagram.)

xxvi. The extension or shortening of the sides of an angle does not change the magnitude of the angle.

50. **A Theorem** is a truth which is made apparent by a course of reasoning or argument. This argument is called a *Demonstration.*

51. Every theorem consists of two distinct parts, either expressed or implied; viz. the *hypothesis* and *conclusion.* The conclusion is the part to be proven, and the demonstration is undertaken only upon the ready granting of the conditions expressed in hypothesis; *e.g.:*

Hyp. If two parallel lines be crossed by a transversal,
Con. the alternate interior angles are equal.

52. In demonstrating the theorems in this book the pupil should first *analyze* the theorem and write it after the above model. For instance, let us analyze the following theorem; viz.:

A perpendicular is the shortest line from a point to a straight line.

Analyzed and written according to our model, this theorem would read as follows:

Hyp. If from a given point to a given straight line a perpendicular and other lines be drawn,
Con. the perpendicular is the shortest one of those lines.

53. The *converse* of a theorem is another theorem in which the hypothesis of the first becomes the conclusion of the second, and the conclusion of the first becomes the hypothesis of the

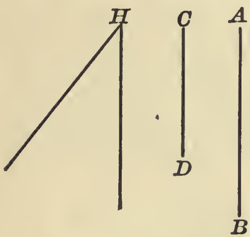
second. For example, the converse of the theorem mentioned in Sec. 51 would read as follows, viz.:

Hyp. If two straight lines in the same plane be crossed by a transversal so as to make the alternate interior angles equal,

Con. these two straight lines will be parallel.

54. **A Corollary** is a theorem which is easily proved from a preceding one with which it is closely associated. The author deems this classification of theorems unnecessary, and so the term will be seldom used in this book.

55. **A Scholium** is a remark made upon one or more preceding propositions pointing out their application or limitation.



56. **A Problem**, in geometry, is the required construction of a geometrical figure from stated conditions or data; *e.g.:*

It is required to construct the triangle which has for two of its sides *AB* and *CD*, and the angle *H* included between these two sides.

57. **A Postulate** is a self-evident problem, or a construction to the possibility of which assent is requested without argument or evidence.

Both theorems and problems are commonly designated as propositions.

58. POSTULATES.

Before we can accomplish the demonstration of a theorem in geometry, the following postulates must be granted, viz.:

i. A straight line can be drawn from one point to any other point.

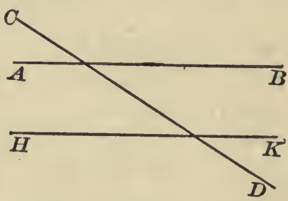
ii. A straight line can be extended to any length, or terminated at any point.

iii. A circle may be described about any point as a center and with any radius.

iv. Geometrical magnitudes of the same kind may be added, subtracted, multiplied, and divided.

v. A geometrical figure may be conceived to be moved at pleasure without changing its size or shape.

59. The term *postulate* may also be used, and in this book is so used, to designate the first step toward the demonstration whereby it is requested that certain conditions be admitted as true, for the basis of the argument, and which begins with "Let," etc., or "Let it be granted that," etc. It requests assent to the general conditions implied in the hypothesis, with special reference to a particular diagram, *e.g.* in beginning the demonstration of the theorem given in Sec. 51 we should say, "Let AB and HK be two parallel straight lines crossed by the transversal CD."



This is the *postulate; i.e.* it matters not whether the material lines AB and HK are actually straight or actually parallel; they stand for straight lines and parallel lines, and the argument is just as conclusive when based upon their *supposed* parallelism as it would be if we had positive knowledge that those two identical lines were parallel.

60. The demonstration of the following theorems should be written by the pupils, with an occasional oral exercise participated in by the entire class, each member in turn contributing a single link in the chain of argument.

61. In order to save time for the pupil in writing and the instructor in correcting, the author, having used them, recommends the use of the following list of symbols and abbreviations, as well as such others as the instructor and pupils may agree upon:

SYMBOLS.

$+$	plus.
$-$	minus.
$\times$	multiplied by.
$=$	equals, or is equal to.
$\therefore$	therefore, or hence.
$\parallel$	parallel.
$\parallel$s	parallels.
$\angle$	angle.
$\angle$s	angles.
rt. $\angle$ or $\angle$r. .	right angle.
rt. $\angle$s or $\angle$r.'s	right angles.
$\triangle$	triangle.
$>$	is greater than.
$<$	is less than.
$\equiv$ or $\asymp$	equivalent to.
$\odot$	circle.
$\odot$s	circles.
$\triangle$s	triangles.
rt. $\triangle$	right triangle.
rt. $\triangle$s	right triangles.
$\perp$	perpendicular.
$\perp$s	perpendiculars.
$\square$	parallelogram.
$\square$s	parallelograms.

ABBREVIATIONS.

Adj. . .	adjacent.
Alt. . . .	alternate.
Ax. . . .	axiom.
Comp. .	complementary.
Con. . .	conclusion.
Cons. . .	construction.
Def. . .	definition.
Dem. . .	demonstration.
Dist. . .	distance.
Ext. . .	exterior.
Hyp. . .	hypothesis.
Iden. . .	identical.
Int.	interior.
Line	straight line.
Opp.	opposite.
Post.	postulate.
Prob.	problem.
Pt.	point.
Quad.	quadrilateral.
St.	straight.
Sug.	suggestion.
Sup.	supplementary.
Trans.	transversal.
Vert.	vertical.
Q.E.D. . . .	*Quod erat demonstrandum.*
Q.E.F. . . .	*Quod erat faciendum.*

The last two expressions are in Latin, and mean, respectively, "which was to be demonstrated" or "proven," and "which was to be performed" or "done." The former is placed at the close of the demonstration of every theorem to indicate that the required proof has been completed, while

the latter is placed at the close of the work of every problem to indicate that the required construction has been accomplished.

62. The pupil must remember that *every statement* in geometrical demonstration must be "backed up" or substantiated by giving as authority definitions, axioms, and previously established truths, and *every unsupported statement*, whether from instructor or fellow-pupil, should be promptly challenged.

63. The author here wishes to emphasize this caution to both instructor and pupil, based on several years' experience in the class-room, viz.: *always employ the most unfavorable diagram.* Care in this particular will save many an inadvertent error and prevent the *assumption* of conditions unwarranted by the hypothesis.

THEOREMS.

64. If two angles are equal, their complemental angles are also equal.

65. If an angle is complemental to each of two other angles, these other two angles are equal.

66. If two angles are equal, their supplemental angles are also equal.

67. If an angle is supplemental to each of two other angles, these other two angles are equal.

68. If two supplemental angles are equal, each is a right angle.

69. If one straight line intersects another, the vertical angles are equal.

Sug. Consult 67.

70. If a line bisect an angle, and a perpendicular be drawn to this bisector through the vertex, the angles which this perpendicular makes with the sides of the angle are equal.

71. If two straight lines intersect each other, the sum of the four angles equals four right angles.

72. The sum of all the *consecutive* angles formed at one point in a straight line and on one side of it equals two right angles.

Sug. Draw a perpendicular, or consult 24.

73. If any number of lines be drawn from the same point, the sum of all the *consecutive* angles equals four right angles.

Sug. Extend one of the lines from the point.

74. If one of the angles formed by the intersection of two straight lines is a right angle, the other three are also right angles.

Sug. Consult 69 and 24.

75. If two adjacent supplemental angles are bisected, the bisectors are perpendicular to each other.

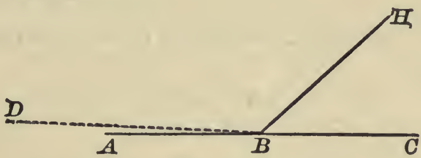
Sug. Prove that the angle which they form with each other is a right angle.

76. **If the bisectors of two adjacent angles are perpendicular to each other, the two angles are supplemental.**

77. If one of two adjacent supplemental angles be bisected, and a perpendicular be drawn to this bisector from the common vertex, this perpendicular will bisect the other angle.

78. **If two adjacent angles are supplemental, their exterior sides form one straight line.**

Post. Let ABH and HBC be 2 adj. $\angle$s, and also let $\angle ABH + \angle HBC = 2$ rt. $\angle$s.



To Prove. AB is the extension of BC.

Cons. Draw BD to represent the extension of BC.

Dem.

$\angle ABH + \angle HBC = 2$ rt. $\angle$s. (?)

$\angle DBH + \angle HBC = 2$ rt. $\angle$s. (?)

$\therefore \angle ABH = \angle DBH$; (?)

$\therefore AB$ must coincide with BD. (?)

$\therefore AB$ is the extension of BC. (?)

i.e. AB and BC form one line.

Q.E.D.

79. If two vertical angles be bisected, the bisectors will form one line.

Sug. Consult 72 and 78, or draw a bisector of one of the other angles, and consult 76 and 25.

80. **If a line, bisecting one of two vertical angles, be extended from the vertex, this extension will bisect the other angle.**

Sug. Consult 69 and Ax. iii., or draw a perpendicular from the vertex, and consult 77.

TRIANGLES.

81. A Triangle is a plane figure bounded by three straight lines, and consequently has three angles. It is sometimes called a *trigon.*

82. Triangles are named,

i. With reference to the character of their angles.

ii. With reference to the relations between their sides.

i. If a triangle has all its angles *acute*, it is called an *acute triangle.*

If it has one *obtuse* angle, it is called an *obtuse triangle.*

If it has one *right* angle, it is called a *right triangle.*

ii. If all its sides are *equal*, it is called an *equilateral triangle.*

If two of its sides are *equal*, it is called an *isosceles triangle.*

If no two of its sides are *equal*, it is called a *scalene triangle.*

83. The term *isosceles* is used to designate the fact that two sides are equal without any reference to the third side, even though it may then be known, or afterward ascertained, that the sides are all equal.

84. If the angles of a triangle are all equal, the triangle is said to be *equiangular.*

If the three angles of one triangle are equal respectively to the three angles of another, the two triangles are said to be *mutually equiangular.*

85. If the sides of one triangle are equal respectively to the three sides of another, the two triangles are said to be *mutually equilateral.*

86. The corresponding sides and angles of two triangles are those that are similarly situated, and are called *homologous.* If the triangles are equal, the *homologous sides* are those that

are opposite the equal angles; and conversely, the *homologous angles* are those that are opposite the equal sides.

87. The sum of the three sides of a triangle is called the *perimeter.*

In a right triangle, the side opposite the right angle is called the *hypothenuse,* and the other two sides the *legs.*

The side on which a triangle is supposed to rest is called the *base.* In general, any side of a triangle may be considered the base; but in an isosceles triangle the *third* side, and in a right triangle one of the *legs,* is generally the base.

The angle opposite the base is called its *vertical* angle; and its vertex, the vertex of the triangle.

The perpendicular distance from vertex to base, or base extended, is called the *altitude* of a triangle.

The line drawn from the vertex of a triangle to the middle point of the base is called a *median.*

88. If two or more lines meet or intersect one another at a common point, they are said to be *concurrent,* and if two or more points lie in the same straight line they are said to be *collinear.*

THEOREMS.

89. Any side of a triangle is less than the sum of the other two.

90. Any side of a triangle is greater than the difference of the other two.

Sug. Consult 89 and Ax. vii.

91. If lines be drawn from a point within a triangle to the ends of either side, the sum of these two lines is less than the sum of the other two sides of the triangle.

Sug. Produce one of the lines until it meets a side of the triangle, then consult 89 and Ax. xvi.

92. If two sides and included angle of one triangle are equal respectively to two sides and included angle of another, the two triangles are equal.

Sug. Superpose one triangle upon the other so that they will have a common vertex and a common side, then prove the congruence of the other parts. See 37 and 38.

93. **If two sides of a triangle are equal, the angles opposite those sides are also equal.**

Sug. Draw a line bisecting the vertical angle, then compare the triangles by 92.

94. **If two angles and included side of one triangle are equal respectively to two angles and included side of another, the two triangles are equal.**

Sug. Use the same method as in 92.

95. **If two triangles are mutually equilateral, they are equal.**

Sug. Apply one to the other as in Fig. I, and consult 93 and 92, or, as in Fig. II, and prove that point *C* must fall on point *H*.

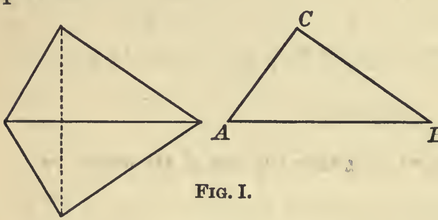


Fig. I.

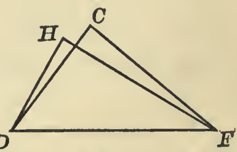


Fig. II.

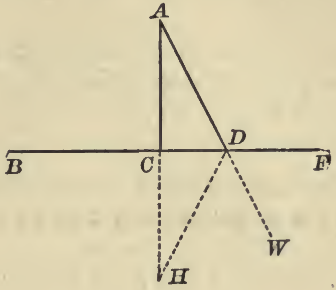
96. **If two angles of a triangle are equal, the sides opposite those angles are also equal.**

Sug. Draw a line from the vertex to the center of the base and compare the triangles.

97. **If two angles of a triangle are unequal, the side opposite the larger angle is longer than the side opposite the lesser.**

Sug. Take a part of the larger angle equal to the smaller angle and have them both in the same triangle, then consult 96 and 89.

98. **From a point to a straight line only one perpendicular can be drawn to that line, or if others are drawn they will all coincide.**



Post. Let AC be $\perp$ to BF and AD any other line from A to BF.

To Prove. AD is oblique to BF.

Cons. Prolong AC from C making $CH = AC$, join HD and extend AD from D.

Dem.

$\triangle ACD = \triangle CDH.$ (?)

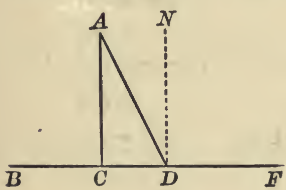
$\therefore \angle ADC = \angle CDH.$ (?)

$\angle CDH < \angle CDW.$ (?)

etc.

99. **A perpendicular is the shortest line that can be drawn from a point to a straight line.**

Cons. Draw $DN \perp$ to BF.



Dem. AD is oblique to BF. (?)

$\therefore$ ADC is an acute $\angle$. (?)

$\angle ACD = \angle CDN.$ (?)

$\angle CDN > \angle ADC.$ (?)

$\therefore \angle ACD > \angle ADC.$ (?)

$\therefore AC < AD.$ (?) Q.E.D.

100. If a perpendicular bisect a line, the lines drawn from any point in the perpendicular to the ends of the line are equal; *i.e.* the point is *equidistant* from the ends of the line.

Sug. Consult 92.

101. If each of two points be equidistant from the ends of a given line, the line passing through these points will be the perpendicular bisector of the given line.

Sug. Consult 95 and 92.

102. If a perpendicular bisect a line, and lines be drawn from any point outside this perpendicular to the ends of the line, the line that crosses the perpendicular is the longer.

Sug. Draw a line from the point of intersection so as to utilize 96 and 89.

103. Every point which is equidistant from the ends of a given line is in the perpendicular bisector of that line.

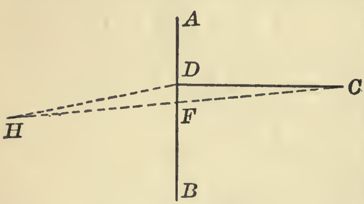
104. Only two equal lines can be drawn from a point to a straight line.

105. The shortest line from a point to a line is perpendicular to the line.

Post. Let CD be the shortest line from C to AB.

To Prove. CD is $\perp$ to AB.

Dem. CD is either $\perp$ to AB or it is not. Let us suppose it



is not. Then draw $CF \perp$ to AB and produce it from F, making $FH = FC$, and join HD.

$$HD + DC > HF + FC. \quad (?)$$

$$HD = DC. \quad (?)$$

$$HF = FC. \quad (?)$$

$$\therefore DC > FC. \quad (?)$$

But $$DC < FC. \quad (?)$$

$\therefore$ we have arrived at a result which contradicts the *Hyp.* etc. Q.E.D.

106. **If a perpendicular divide a line unequally and lines be drawn from any point in the perpendicular to the ends of the line, the line drawn to the end the farther from the perpendicular is the longer.**

Sug. Draw the ⊥ bisector of the line, and draw another line so as to utilize 96 and 97.

107. **If two sides of a triangle are unequal, the angle opposite the longer side is greater than the angle opposite the shorter.**

Sug. Draw the ⊥ bisector of the third side, and connect the point of intersection and end of the third side. Then consult 92 and 93.

108. **If two lines are perpendicular to the same line, they are parallel.**

Sug. Two possible relations. One of them contradicts a previous theorem.

109. **If a line is perpendicular to one of two parallel lines, it is perpendicular to the other also.**

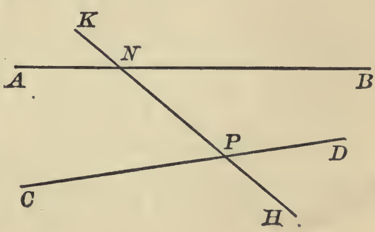
Sug. From the point of intersection with the latter draw a ⊥ to the perpendicular, then consult 108 and Ax. xxiv.

110. **If two lines are parallel to the same line, they are parallel to each other.**

Sug. Draw a ⊥ to one of the lines, and consult 109.

TRANSVERSALS.

111. When two straight lines in the same plane are crossed by another, the latter is called a *transversal*. In the figure, which line is the *transversal?*



How many angles are formed? Certain ones of the above angles are termed *interior angles.* Name them. Why are they called *interior angles?*

What name would you

give to the others to distinguish them from those already named? Why? How many pairs of *vertical* angles? Name them.

How many pairs of *adjacent* angles? Name them.

Two angles situated on opposite sides of the transversal, either both interior or both exterior, and not adjacent, are called *alternate* angles.

How many pairs of *alternate interior* angles? Name them.

How many pairs of *alternate exterior* angles? Name them.

Two non-adjacent angles, of which one is interior and the other exterior, and both on the same side of transversal, are called technically *exterior interior* angles.

How many pairs of *exterior interior* angles? Name them.

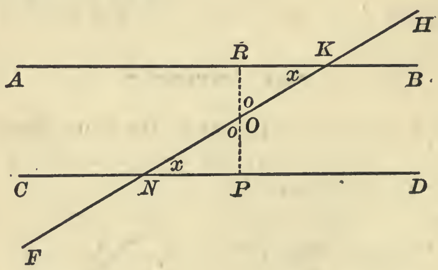
Angles on the same side of the transversal are called *lateral angles.*

How many pairs of *lateral interior* angles? Name them.

How many pairs of *lateral exterior* angles? Name them.

112. If two lines in the same plane are crossed by a transversal making the alternate interior angles equal, the two lines are parallel.

Post. Let AB and CD be two lines crossed by trans. FH, making $\angle\ CNK = \angle\ NKB$.

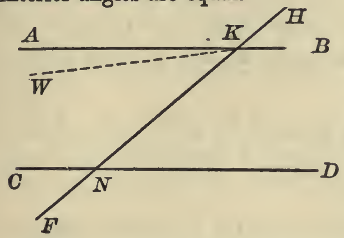


To Prove. AB is $\parallel$ to CD.

Cons. Bisect NK and through its center O draw $RP \perp$ to CD.

Dem. Compare the $\triangle$s, and consult 108.

113. If two parallel lines are crossed by a transversal, the alternate interior angles are equal.



Post. ————.

To Prove. ———.

Cons. Draw WK, making $\angle WKN = \angle KND$.

Dem.

$\angle WKN = \angle KND$. (?)

$\therefore WK$ is $\parallel$ to CD. (?)

AB is $\parallel$ to CD. (?)

$\therefore AB$ coincides with WK or contradicts Ax. xxiv, etc.

Q.E.D.

114. If two parallel lines are crossed by a transversal, the exterior interior angles are equal.

115. Converse of 114.

116. If two parallel lines are crossed by a transversal, the alternate exterior angles are equal.

117. Converse of 116.

118. If two parallel lines are crossed by a transversal, the lateral interior angles are supplemental.

119. Converse of 118.

120. If two parallel lines are crossed by a transversal, the lateral exterior angles are supplemental.

121. Converse of 120.

122. If two lines in the same plane be crossed by a transversal making the alternate interior angles unequal, the lines will meet, if sufficiently prolonged, on that side of the transversal on which is the smaller interior angle.

Sug. Draw a line through the vertex of the larger angle making an angle equal to the smaller so as to utilize 112 and Ax. xxv.

123. If two lines in the same plane be crossed by a transversal making the sum of the lateral interior angles less than two right angles, the lines will meet, if sufficiently extended, on that side of the transversal on which the sum of the angles is less than two right angles. *Sug.* Consult 122.

124. If two parallel lines are crossed by a transversal, the lines bisecting either pair of alternate interior angles are parallel.

125. If two parallel lines are crossed by a transversal, the lines bisecting the lateral interior angles are perpendicular to each other.

Sug. Consult 124, 75, and 109.

126. The sum of the three angles of a triangle equals two right angles. *Sug.* Consult 72 and 113.

127. If two angles of one triangle are equal respectively to two angles of another, their third angles are equal.

128. If the sum of two angles of a triangle is equal to the third, the latter is a right angle.

129. If one side of a triangle be produced, the exterior angle equals the sum of the two interior angles not adjacent to it.

130. If one of the equal sides of an isosceles triangle be extended at the vertex, the exterior angle is double either of the base angles.

131. If the legs of a right triangle are equal respectively to the legs of another, the triangles are equal.

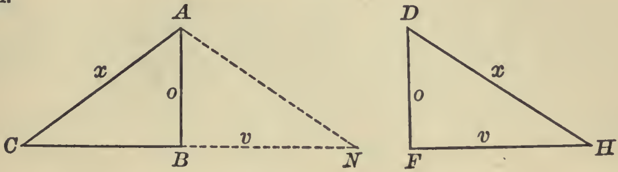
132. If the hypothenuse and an acute angle of one right triangle are equal respectively to the hypothenuse and acute angle of another, the triangles are equal.

Sug. Consult 127 and 94.

133. If a leg and acute angle of one triangle are equal respectively to a leg and homologous acute angle of another, the triangles are equal.

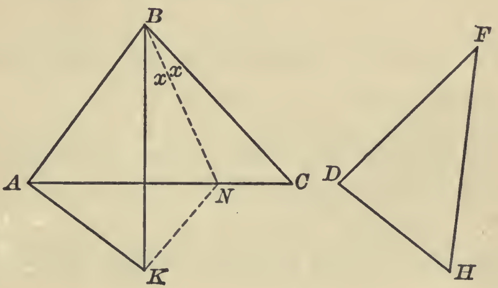
Sug. Case I when the equal angles are adjacent to the legs.
Case II when the equal angles are opposite the legs.

134. If the hypothenuse and leg of one right triangle are equal respectively to the hypothenuse and leg of another, the triangles are equal.



Cons. Extend CB from B making $BN = FH$, then consult 131, 103, and 95.

135. If two sides of one triangle are equal respectively to two sides of another, but the included angle of one greater than the included angle of the other, then the third side of that triangle having the greater included angle is longer than the third side of the other.



Post. Let ABC and DFH be 2 $\triangle$ with sides $AB = DF$ and $BC = FH$, and $\angle ABC$ greater than $\angle F$, and let AB be the side *not greater* than BC.

To Prove $AC > DH$.

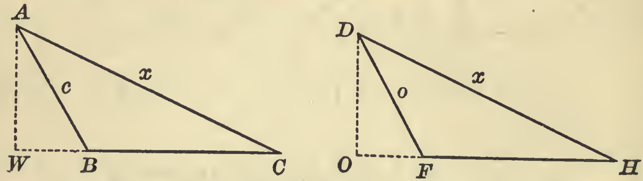
Cons. Draw AK and BK equal respectively to DH and FH. Draw BN bisecting $\angle KBC$ and join KN.

Dem. Compare $\triangle$ ABK and DFH, also $\triangle$ KBN and BCN, then consult 89.

136. If two sides of one triangle are equal respectively to two sides of another, but their third sides unequal, then the angle opposite the longer third side is greater than the angle opposite the shorter.

Sug. Three possible relations.

137. If two sides and obtuse angle opposite one, of one triangle, be equal respectively to two sides and obtuse angle opposite one, of another, the two triangles are equal.



Cons. Produce BC and FH from B and F. Draw ⊥s AW and DO and compare △s.

138. If two triangles are equal, their homologous altitudes are equal.

139. If a triangle is equilateral, it is also equiangular.

140. Converse of 139.

141. If a line bisect the vertical angle of an isosceles triangle, it will be perpendicular to, and bisect, the base.

142. If a perpendicular bisect the base of an isosceles triangle, this perpendicular will pass through the vertex and bisect the vertical angle.

143. If a line be drawn from the vertex of an isosceles triangle perpendicular to the base, it will bisect both the base and vertical angle.

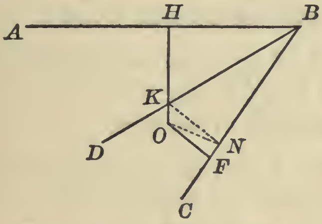
144. If a line be drawn from the vertex of an isosceles triangle to the center of the base, it will be perpendicular to the base and bisect the vertical angle.

145. The three altitudes of an equilateral triangle are equal.

146. If a line bisect an angle, the perpendiculars from any point in the bisector to the sides of the angle are equal; *i.e.* the point is equidistant from the sides of the angle.

147. If equal perpendiculars be drawn from any point to the sides of an angle, the line which joins this point to the vertex will bisect the angle; *i.e.* if a point be equidistant from the sides of an angle, etc.

148. If a line bisect an angle and perpendiculars be drawn from any point outside this bisector to the sides of the angle, the perpendicular that crosses the bisector is the longer.



Post. Let ABC be an $\angle$ bisected by DB, etc.

To Prove. $OH > OF$.

Cons. Draw $KN \perp$ to BC and join ON.

Sug. Compare OF with ON, and ON with $OK + KN$, etc.

149. The three bisectors of the angles of a triangle are concurrent.

Sug. From the point of intersection of *two* bisectors draw a line to the third vertex and also $\perp$s to the three sides. Prove the former a bisector by a comparison of triangles, or consult 146 and 147.

150. The perpendicular bisectors of the sides of a triangle are concurrent.

Sug. From the point of intersection of *two* of the $\perp$s draw a line to the center of the third side and prove this line a $\perp$.

151. The perpendiculars from the three vertices of a triangle to the opposite sides are concurrent.

Sug. Through each vertex draw a line $\parallel$ to the opposite side and consult 150.

ADVANCE THEOREMS.

The following theorems are to be demonstrated entirely independent of one another.

They vary as to difficulty, and are given for the purpose of affording additional practice for the pupil if the instructor should deem it necessary, as well as offering further scope for the enthusiasm of the pupil.

152. If one angle of an isosceles triangle is two thirds of a right angle, the triangle is equilateral.

153. If the base angles of an isosceles triangle be bisected, the bisectors will form, with the base, an isosceles triangle.

154. If one of the equal sides of an isosceles triangle be extended at the vertex and the exterior angle thus formed be bisected, this bisector will be parallel to the base.

155. If one of the equal sides of an isosceles triangle be extended at the vertex and a line be drawn through the vertex parallel to the base, this line will bisect the exterior angle.

156. If one of the equal sides of an isosceles triangle be extended at the vertex, making the extension equal to the side, and its extremity joined to the end of the base, this line will be perpendicular to the base.

157. If each of the base angles of an isosceles triangle be double the vertical angle, a line bisecting either of the former will divide the triangle into two isosceles triangles.

158. If two triangles have two unequal sides and an angle opposite the greater of one, equal respectively to two unequal sides and an angle opposite the greater of the other, the two triangles are equal.

Sug. From the vertices of the angles formed by the given sides draw ⊥s to the third sides or their extensions, and compare the right angles.

159. If two angles of an equilateral triangle be bisected and lines be drawn through the point of intersection parallel to the sides, the sides will be trisected.

160. The two perpendiculars from the extremities of the base of an isosceles triangle to the equal sides are equal.

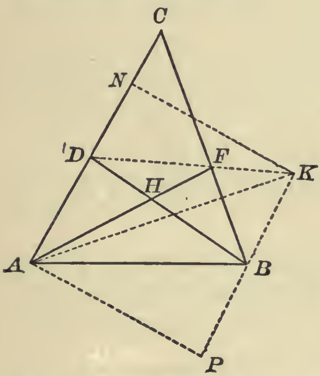
161. Converse of 160.

162. The medians drawn to the equal sides of an isosceles triangle are equal.

163. Converse of 162.

164. The bisectors of the base angles of an isosceles triangle, terminating in the equal sides, are equal.

165. Converse of 164.



Post. Let $AF = DB$ and bisect $\angle$s DAB and ABF.

To Prove. $AC = BC$.

Cons. Make

$\angle BDK = \angle FAB$,

and $\angle DBK = \angle AFB$.

Join AK. Extend BK and draw $AP \perp$ to KP and $KN \perp$ to AC.

Dem. $\triangle DBK = \triangle FAB$.

$\therefore AB = DK$.

$\angle AHB = \angle ADH + \angle DAH$.

$\therefore \angle AHB = \angle ADK$.

$\angle AHB = \angle HFB + \angle HBF$.

$\therefore \angle AHB = \angle HBK + \angle HBA$.

$\therefore \angle AHB = \angle ABK$.

$\therefore \angle ADK = \angle ABK$.

$\therefore \angle NDK = \angle ABP$.

$\therefore \triangle NDK = \triangle ABP$.

$\therefore ND = BP$ and $NK = AP$.

$\therefore \triangle NAK = \triangle AKP$.

$\therefore NA = KP$.

$\therefore AD = KB$.

But $FB = KB$.

$\therefore AD = FB$.

$\therefore \triangle ADB = \triangle AFB$.

$\therefore \angle ABD = \angle FAB$.

$\therefore \angle DAB = \angle ABF$.

$\therefore AC = BC$. Q.E.D.

166. The two perpendiculars from the middle point of the base of an isosceles triangle to the equal sides are equal.

167. Converse of 166.

168. If perpendiculars be drawn to the equal sides of an isosceles triangle through the extremities of the base and meeting below the base, the triangle thus formed is isosceles.

169. If a median bisects an angle of a triangle, this triangle is isosceles.

170. If the vertical angle of an isosceles triangle is a right angle, the altitude is equal to one half the base.

171. If a line be drawn parallel to the base of an isosceles triangle and meeting the other sides, the triangle thus formed is isosceles.

172. If a line bisecting an exterior angle of a triangle is parallel to a side, this triangle is isosceles.

173. If two isosceles triangles have the base and vertical angle of one equal to the base and vertical angle of the other, the two triangles are equal.

174. If, from any point in the bisector of an angle, a line be drawn parallel to one side and terminating in the other, the triangle thus formed is isosceles.

175. If a perpendicular be drawn to the base of an isosceles triangle intersecting one of the equal sides and produced to meet the extension of the other, the triangle thus formed is isosceles.

176. If the base angles of an isosceles triangle be bisected, the line joining the points of intersection of these bisectors with the sides of the triangle is parallel to the base.

177. If medians be drawn to the equal sides of an isosceles triangle, the line joining the points of meeting is parallel to the base.

178. If perpendiculars be drawn to the equal sides of an isosceles triangle from the ends of the base, the line joining the points of meeting is parallel to the base.

179. If one side of a triangle be produced in both directions, the sum of the exterior angles exceeds two right angles.

180. If, from a point within a triangle, lines are drawn to the ends of one side, the angle formed by these lines is greater than the angle formed by the other two sides of the triangle.

181. If the two base angles of a triangle are bisected and through the point of intersection a line be drawn parallel to the base and terminating in the sides, this line is equal to the two segments of the sides between the two parallels.

182. The bisectors of the vertical angle of a triangle and of the angles formed by extending the sides below the base are concurrent, and the point of meeting is equidistant from the base and sides extended.

183. If a line be drawn from one extremity of the base of an isosceles triangle perpendicular to the opposite side, the angle formed by this line and the base is one half the vertical angle.

184. If an angle be bisected and a perpendicular be drawn to this bisector terminating in the sides of the angle, the triangle thus formed is isosceles.

ANGLE MEASUREMENT.

185. The most common unit for measuring the magnitude of angles is the ninetieth part of a right angle, called a *degree.* The degree is subdivided into sixty equal parts called *minutes,* and the latter also into sixty equal parts called *seconds.* By this means small fractions of a degree may be expressed in minutes and seconds. These units are indicated by the following symbols, viz.:

$^\circ$ for degrees, $'$ for minutes, and $''$ for seconds; *e.g.* $75^\circ\ 20'\ 34''$.

EXERCISES.

186. 1. How many degrees in a right angle?

2. What is the complement of a right angle?

3. What is the supplement of a right angle?

4. How many degrees in all the consecutive angles formed at one point on one side of a straight line?

5. How many degrees in all the consecutive angles formed by any number of lines meeting at one point?

6. What is the complement of 45°; 30°; 1°; $80^\circ\ 59'\ 59''$; $27^\circ\ 37'\ 47''$?

7. What is the supplement of each of the above-named angles?

8. How many degrees in an angle that is double its complement?

9. How many degrees in an angle that is four times its supplement?

10. What angle, in degrees, do the hour and minute hands of a clock form at 2 o'clock? At 3 o'clock? At 5 o'clock?

11. If one of the angles formed by two straight lines crossing each other be 120°, what are the values of the other angles?

12. If two complemental adjacent angles be bisected, how many degrees in the angle formed by the bisectors?

13. What is the complement of $37° 44' 51''$?

14. What is the supplement of $104° 33' 21''$?

15. One half a right angle is what part of three right angles? Of two right angles?

16. One fifth of two right angles is what part of one right angle? Of four right angles?

17. How many degrees in an angle that is one fifth of its complement?

18. How many degrees in an angle that is one third of its supplement?

19. Two lines are crossed by a transversal, making one pair of lateral exterior angles $54° 48' 32''$ and $125° 11' 28''$ respectively; how are the two lines related to each other?

20. If one of two complemental angles is acute, what kind of an angle must the other be?

21. If one of two supplemental angles is acute, what kind must the other one be?

22. If two angles of a triangle be known, how can the third be found?

23. Two angles of a triangle are $34° 28' 42''$ and $29° 44' 56''$ respectively; find the value of the third angle.

24. If one angle of a triangle be a right angle, what relation exists between the other two?

25. How many right angles can there be in a triangle? How many obtuse angles?

26. If the three angles of a triangle are all equal, what is the value of each angle in terms of a right angle? In degrees?

27. If one acute angle of a right triangle is $27° 38' 50''$, what is the value of the other acute angle?

28. Find values of the three angles of a triangle, if the second is three times the first and the third is two times the second.

29. In a right triangle one of the acute angles is $17° 40'$ greater than the other; find the values of both acute angles.

30. Two adjacent angles are $32° 20' 40''$ and $57° 39' 20''$; what is the relation of the exterior sides to each other?

31. Two adjacent angles are $68° 35' 52''$ and $111° 24' 8''$; what is the relation of the exterior sides to each other?

32. What part of its complement is an angle that is one fourth of its supplement? One fifth? One ninth?

33. What part of its supplement is an angle that is four times its complement? Five times? Six times?

34. Four consecutive angles are formed at one point on one side of a straight line; three of these angles are $22° 22' 22''$; $44° 44' 44''$; and $55° 55' 55''$; what is the size of the other one?

35. An exterior angle of a triangle is $76° 15' 12''$, and one of the interior non-adjacent angles is $38° 26' 10''$; how many degrees in the remaining angles?

36. The vertical angle of an isosceles triangle is $35° 45' 55''$; what is the size of the base angles?

37. A triangle is both isosceles and right; how many degrees in each angle?

38. How many degrees in each angle of an isosceles triangle whose vertical angle is one fourth one of the base angles? One seventh? One half?

39. How many degrees in each angle of an isosceles triangle if one of the base angles is twice the vertical angle? Four times? Seven times?

40. How many degrees in each angle of an isosceles triangle if the vertical angle is twice one of the base angles? Three times? Four times? Seven times?

QUADRILATERALS.

187. If two lines in the same plane be crossed by two transversals, a figure of four sides may be formed, which is called a *quadrilateral.* Hence a quadrilateral may be defined as a plane figure bounded by four straight lines. A quadrilateral is also called a *tetragon.*

188. If each of the two pairs of lines is *parallel,* the quadrilateral thus formed is called a *parallelogram.*

Define, then, a parallelogram.

189. If a parallelogram have all its sides *equal,* it is called a *rhombus.*

190. If its angles are right angles, it is called a *rectangle.*

191. If it is both a rhombus and a rectangle, it is called a *square.*

192. If a quadrilateral have only two of its sides parallel, it is called a *trapezoid.*

If no two of its sides are parallel, it is called a *trapezium.*

193. A parallelogram whose angles are oblique and adjacent sides unequal is sometimes called a *rhomboid.*

194. A rectangle whose adjacent sides are unequal is sometimes called an *oblong.*

195. The line which joins two opposite vertices of a quadrilateral is called a *diagonal.*

196. The side upon which a parallelogram is conceived to rest, and the side opposite the latter, are termed, respectively, the *lower* and *upper bases.*

197. The parallel sides of a trapezoid are always considered as its bases, the other two sides its *legs,* while the line bisecting the legs is called the *median,* and the line which bisects the bases is called the *transverse median.*

198. The *altitude* of either a parallelogram or trapezoid is the perpendicular distance between its bases.

199. The sum of the four sides of a quadrilateral is called its *perimeter*.

200. If the *legs* of a trapezoid are equal, it is called an *isosceles trapezoid.*

THEOREMS.

201. The sum of the four angles of a quadrilateral equals four right angles.

202. Either diagonal divides the parallelogram into two equal triangles.

Sug. Consult 113 and 94.

203. The opposite sides of a parallelogram are equal.

204. Converse of 203.

Sug. Draw one diagonal and consult 95 and 112.

205. The opposite angles of a parallelogram are equal.

206. Converse of 205.

Sug. Consult 201 and 119.

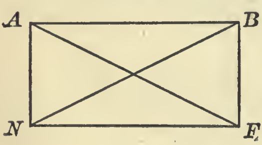
207. If two sides of a quadrilateral are equal and parallel, the quadrilateral is a parallelogram.

208. If a quadrilateral have two opposite sides equal and two opposite angles equal, this quadrilateral is a parallelogram.

209. The two diagonals of a parallelogram bisect each other.

210. Converse of 209.

211. If one angle of a parallelogram be a right angle, the other three angles are right angles, and the parallelogram is therefore a rectangle.



212. The diagonals of a rectangle are equal.

Sug. Compare any two triangles that have one side of the rectangle for a common side.

213. Converse of 212.

214. The diagonals of a rhombus are perpendicular to each other.

215. Converse of 214.

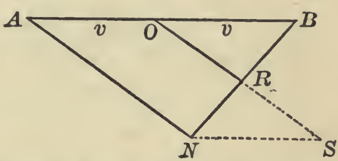
216. The diagonals of a rhombus bisect the angles.

217. If one diagonal of a parallelogram bisects one of its angles, this parallelogram is a rhombus.

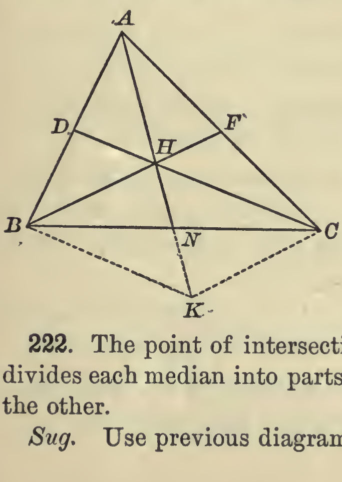
218. If two parallelograms have two sides and their included angle of one equal respectively to two sides and their included angle of the other, the two parallelograms are equal in every respect.

219. If a line, parallel to the base of a triangle, meets the other two sides bisecting one of them, it will also bisect the other and be equal to one half the base.

Sug. Draw $NS \parallel$ to AB and extend OR to meet it. Compare the two triangles and consider the quadrilateral $AOSN$.



220. If a line joins the middle points of two sides of a triangle, it will be parallel to the third side and equal to one half of it.



221. The three medians of a triangle are concurrent.

Sug. In the triangle ABC let CD and BF be two medians, then draw AK through H to meet BK drawn $\parallel$ to DC and join KC. Then consult 219, 220, and 209.

222. The point of intersection of the medians of a triangle divides each median into parts one of which is twice as long as the other.

Sug. Use previous diagram.

223. The line joining the vertex of the right angle to the center of the hypothenuse of a right triangle is equal to one half the hypothenuse.

Sug. Draw a ∥ to one of the legs from the center of the hypothenuse, then consult 219 and compare triangles.

224. **The median of a trapezoid is parallel to its bases and equal to one half their sum.**

Sug. Through one extremity of the median draw a line parallel to one leg, and extend the shorter base to meet it.

ADVANCE THEOREMS.

225. If a quadrilateral have one reflex angle, its exterior explemental angle is equal to the sum of the three interior non-adjacent angles of the quadrilateral.

226. A line parallel to the bases of a trapezoid and bisecting either diagonal is a median.

227. The line which is parallel to the bases of a trapezoid and bisects one leg is a median.

228. If the angles at one base of a trapezoid are equal, the angles at the other base are also equal.

229. If the legs of a trapezoid are equal, the angles which they make with either base are equal, and conversely.

230. If the lesser base of an isosceles trapezoid equals either leg, the diagonals bisect the angles at the longer base.

231. If from any point in the base of an isosceles triangle lines are drawn parallel to the equal sides, the perimeter of the parallelogram thus formed will be equal to the sum of the two equal sides of the triangle.

232. If two opposite angles of a quadrilateral are right angles, the bisectors of the other two angles are parallel.

233. The lines which join the middle points of the sides of a triangle divide the triangle into four equal triangles.

234. If a line be drawn from the middle point of one side of a triangle to meet another side and is equal to one half the base, it will be parallel to the base.

235. If two parallel lines be crossed by a transversal, the bisectors of the interior angles form a rectangle.

236. The lines which bisect the angles of a rhomboid form a rectangle.

237. The lines which join the middle points of the sides of any quadrilateral in succession form a parallelogram.

Sug. Draw a diagonal of the quadrilateral, then consult 220.

238. If the middle points of two opposite sides of a quadrilateral be joined to the middle points of the diagonals, these lines will form a parallelogram.

239. The lines which join the middle points of the sides of a rhombus form a rectangle.

240. The lines which join the middle points of the sides of a square form a square.

241. The lines which join the middle points of the sides of a rectangle form a rhombus.

242. The lines which join the middle points of the sides of an isosceles trapezoid form a rhombus.

243. The median of a trapezoid bisects both diagonals.

Sug. Consult 224 and 219.

244. The diagonals of an isosceles trapezoid are equal, and conversely.

245. If a median of a triangle equals one half the base, this triangle is a right triangle.

246. If a trapezoid be isosceles, the opposite angles are supplemental.

247. If a trapezoid be isosceles, the angles which the diagonals make with either base are equal.

248. The point of intersection of the diagonals of a trapezoid bisects the line parallel to the bases and terminating in the legs.

249. The transverse median of a trapezoid bisects any line parallel to the bases and terminating in the legs.

250. The line which joins the middle points of the diagonals of a trapezoid equals one half the difference of the bases.

251. If from the middle points of two opposite sides of a parallelogram lines be drawn to the vertices of the angles opposite, these lines will trisect the diagonal that joins the other two vertices.

252. If one of the acute angles of a right triangle is double the other, the hypothenuse is double the shorter leg.

253. If from any two points in the base of an isosceles triangle perpendiculars be drawn to the equal sides, the sum of the perpendiculars from one point equals the sum of the perpendiculars from the other point.

254. If from any point in an equilateral triangle perpendiculars be drawn to the sides, the sum of these perpendiculars is constant, and equal to the altitude of the triangle.

255. The medians of any quadrilateral bisect each other.

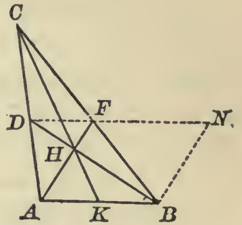
256. The point of intersection of the medians of any quadrilateral bisects the line which joins the middle points of the diagonals.

257. The point of intersection of the diagonals of a parallelogram and the middle points of two opposite sides are collinear.

258. The point of intersection of the diagonals of a parallelogram bisects any line drawn through that point and terminating in the sides.

259. If the bisectors of the angles of a quadrilateral form another quadrilateral, the opposite angles of the latter are supplemental.

260. If two medians of a triangle are perpendicular to each other, the third can be made the hypothenuse, and the other two the legs, of a right triangle.



Cons. Draw DN through F and BN $\parallel$ to AF.

Dem. DBN is a rt. $\triangle$.

$AFNB$ is a parallelogram.

$DN = 3\,DF$, and $CK = 3\,HK$.

Then consult 223.

261. If a diagonal of a quadrilateral bisect two opposite angles, the two diagonals are perpendicular to each other.

262. If two quadrilaterals be mutually equiangular, and the sides, including two of the angles, respectively equal, the two quadrilaterals are equal.

CIRCLES.

263. **A Circle** is a portion of a plane bounded by a curved line, all points of which are equidistant from a point within called the *center.*

264. The bounding line is called the *circumference,* and any portion of the circumference is called an *arc.*

265. **A Radius** (plural *Radii*) is any line from center to circumference. **A Diameter** is any line passing through the center and terminating both ways in the circumference.

266. **A Semicircumference** is one half the circumference, and a **Semicircle** is one half the circle.

267. **A Sector** is that part of a circle included between an arc and the radii drawn to its extremities, and a **Quadrant** is a sector which is one fourth the circle.

268. **A Chord** is any straight line whose extremities are in the circumference.

269. **A Segment** is that portion of the circle included between an arc and the chord which joins its extremities. Every chord, therefore, must divide the circumference into two arcs and the circle into two segments.

270. If the arcs are unequal, they are designated as *major* and *minor* arcs, and the segments as *major* and *minor* segments.

271. The chord is said to *subtend* the arc, and the arc is said to be *subtended* by the chord.

Whenever a chord and its subtended arc are mentioned, the *minor arc* is meant unless it is otherwise specified.

272. If two circles have the same center, they are said to be *concentric.*

273. A central angle is an angle formed by two radii of the same circle.

An inscribed *angle* is an angle formed by two chords, with its vertex in the circumference.

274. *An angle is inscribed in a segment* when its vertex is on the arc of that segment and its sides meet the extremities of the subtending chord.

275. **A Tangent** to a circle is a straight line which has only one point in common with the circumference, however far extended both ways.

276. The *point of contact* of a tangent to a circle is the common point of the tangent and circumference; hence,

All points in a tangent to a circle except the point of contact *lie outside of and beyond the circumference.*

277. Two circumferences are *tangent to each other* when they have one point in common but do not intersect. This point is called the *point of contact*, all other points in either circumference being *outside of and beyond* the other.

278. If one of two tangent circumferences lies within the other, they are said to be tangent *internally;* if it lies without, they are said to be tangent *externally.*

279. **A Secant** is a straight line that intersects a circumference in two points lying partly within and partly without the circle; *e.g.* a chord extended in either or both directions becomes a secant.

280. The term *circle* is also sometimes used to designate a circumference.

281. The distance of a point from a circumference means the distance of that point from the nearest point in that circumference.

282. A triangle or quadrilateral is inscribed in a circle when the vertices of its angles lie on the circumference; *i.e.* when its sides are chords of the circle. In this case the circle is said to be *circumscribed* about the triangle or quadrilateral.

283. A triangle or quadrilateral is *circumscribed* about a circle when its sides are tangents to the circle. In this case the circle is said to be *inscribed* in the triangle or quadrilateral.

284. If the circumference of a circle be made to pass through the vertices of the angles of a quadrilateral, this quadrilateral is said to be *inscriptible.*

285. From the foregoing definitions the following statements are readily deduced which are axiomatic in nature:

All radii of the same circle are equal.

All diameters of the same circle are equal.

A diameter of a circle is double its radius, or the radius is one half the diameter.

If two circles be equal, their circumferences are equal, their radii are equal, and their diameters are equal.

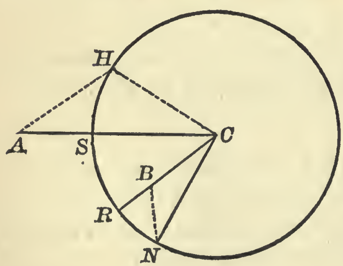
If two circles have equal radii, or equal diameters, or equal circumferences, the circles are equal.

If two equal circles be concentric, their circumferences will coincide.

Should there be any doubt in regard to any one of these, it should be promptly tested by superposing one circle upon the other.

THEOREMS.

286. The point on a circumference nearest to a given point is the point of intersection of the circumference and line joining the given point with the center of the circle.



Sug. There will be two cases,

i. The given point A is outside the circle.

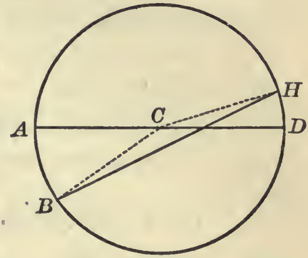
ii. The given point B is within the circle.

Then let H and N be any points on the circumference except the points of intersection S and R.

287. A diameter of a circle is greater than any other chord of that circle.

Post. Let AD be a diameter and BH any other chord.

Sug. Consult 89.

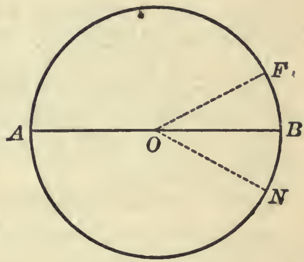


288. A diameter bisects both the circle and its circumference.

Sug. Fold one segment over, using the diameter as an axis.

289. If a chord of a circle bisects the circumference, this chord is a diameter.

Sug. Draw a diameter from one end of the chord and compare arcs.



290. A straight line cannot intersect the circumference of a circle in more than two points.

Sug. Consult 104.

291. If, in the same or equal circles, two central angles be equal, the arcs which their sides intercept will also be equal.

Sch. In this and the following theorems where the expression "same or equal circles" occurs, the demonstration will be more satisfactory if two circles are used.

Sug. Apply one circle to the other.

292. Converse of 291.

293. If, in the same or equal circles, two chords be equal, the arcs which those chords subtend are also equal.

Sug. Draw radii to the extremities of the chords, then consult 95 and 291.

294. Converse of 293.

Sug. Draw radii as above. Consult 292 and 92.

295. If, in the same or equal circles, two central angles be unequal, the arc intercepted by the sides of the greater angle is greater than the arc intercepted by the sides of the lesser angle.

Sug. Take a part of the larger angle equal to the smaller.

296. Converse of 295.

297. If, in the same or equal circles, two chords be unequal, the arc subtended by the greater chord is greater than that subtended by the lesser.

Sug. Draw radii to extremities, then consult 136 and 295.

298. Converse of 297.

Sug. Draw radii as above, then consult 296 and 135, or three possible relations.

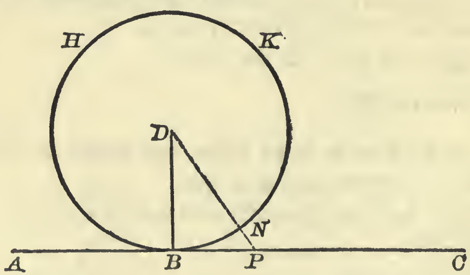
299. If a diameter be perpendicular to a chord, it will bisect the chord, and also the arcs into which the chord divides the circumference.

Sug. Draw radii to extremities of the chord, then consult 134, 291, and 66.

Sch. In the subsequent application of this theorem it will often happen that only half the diameter will be involved, in which case the theorem may be quoted as follows:

If a radius be perpendicular to a chord, it will bisect the chord and also its subtended arc.

300. A straight line that is perpendicular to a radius at its extremity is a tangent to the circle.



Post. Let BHK be a circle, DB a radius, and AC a straight line perpendicular to DB and passing through the point B.

To Prove. That AC is a tangent to the circle BHK.

Now, if we can prove that every point in AC except B is outside the circumference, then AC must be a tangent. (See Def. of *tangent.*)

Cons. From D draw any other line to AC, as DP.

Then P, the extremity of DP, must represent any point in AC except B.

Now $DP > DB$. Why?

But DB is a radius, and if DP is longer than a radius, where must its extremity be?

Hence, etc.

301. If a diameter or radius be drawn to the point of contact of a tangent to a circle, it will be perpendicular to the tangent.

Sug. Prove that the radius is the shortest line from the center to the tangent, and use diagram similar to the preceding.

302. If a perpendicular be drawn to a tangent at the point of contact, this perpendicular, if extended, will pass through the center of the circle.

Sug. Draw a radius to point of contact, then consult 301 and 25.

303. If a line be drawn from the center of a circle perpendicular to a tangent, this line will pass through the point of contact.

304. If a chord and tangent be parallel, the arcs which they intercept are equal.

Sug. Draw a radius to point of contact, then consult 301, 109, and 299.

305. If two chords of a circle be parallel, the arcs which they intercept are equal.

306. If two tangents of a circle be parallel, the arcs which they intercept are equal.

307. The points of contact of two parallel tangents and the center of the circle are collinear.

308. If two tangents have their points of contact the extremities of a diameter, the tangents are parallel.

309. If a radius bisect a chord, it will bisect the arc and be perpendicular to the chord.

310. If a perpendicular bisect a chord, it will, if extended, pass through the center of the circle.

311. If a radius bisect an arc, it will also bisect its subtending chord and be perpendicular to it.

312. The middle points of a chord and its subtended arc and the center of the circle are collinear.

313. The middle points of two parallel chords and the center of the circle are collinear.

314. If two non-intersecting chords intercept equal arcs, they are parallel.

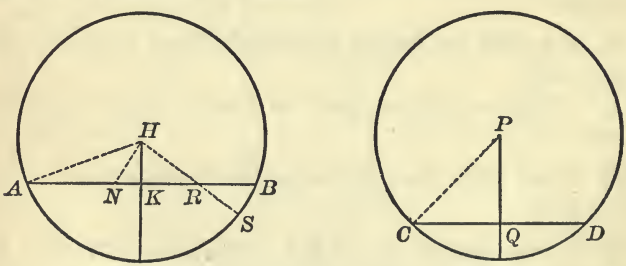
315. Converse of 304.

316. If in the same or equal circles two chords be equal, they are equidistant from the centers.

Sug. Draw radii and compare the triangles.

317. Converse of 316.

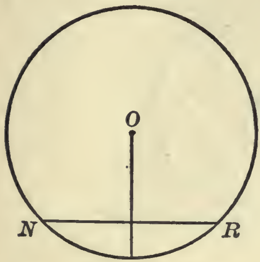
318. If in the same or equal circles two chords be unequal, the lesser chord is the farther from the center.

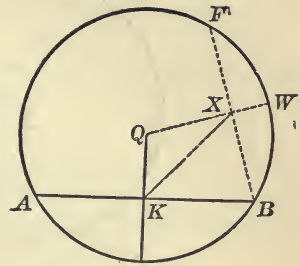


Sug. Make $AR = CD$. Draw HS through R and HN to bisect AR. Compare angles A and C by means of $\triangle$ AHK and CPQ (see 126, 296, etc.), then compare HN and PQ (see 135).

319. Converse of 318.

Sug. Three possible relations, or





Draw chord $BF = NR$, $QW \perp$ to BF, and join KX; then consult 107 and 97.

320. Through any three points that are not in the same straight line, one circumference can be made to pass, and only one; or, if more be drawn, they must coincide.

Sug. Join two pairs of points, then consult 100.

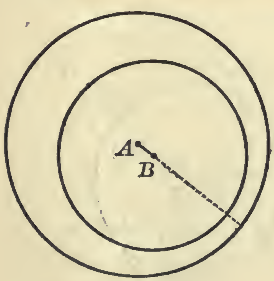
321. If two unequal circles be concentric, the chords of the greater, which are tangents of the less, are equal.

Sug. Consult 317.

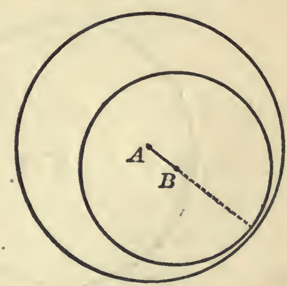
322. Two unequal circles may have five positions relative to each other, viz.:

i. One may lie wholly within the other without contact.

ii. They may be tangent to each other internally.

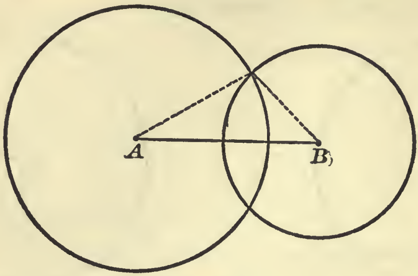


CASE I.



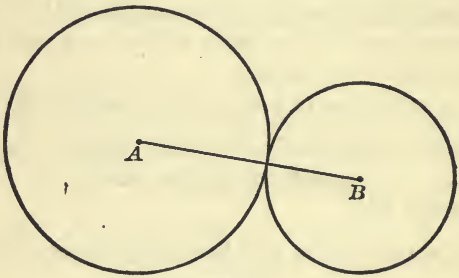
CASE II.

iii. Their circumferences may intersect each other.



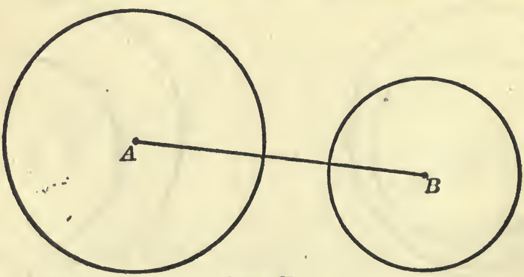
Case III.

iv. They may be tangent to each other externally.



Case IV.

v. One may lie wholly without the other without contact.



Case V.

AB is the line joining the centers. Designating the radius of the larger circle by R and radius of the smaller by r, the pupil should ascertain the relation between AB and $R+r$ and AB and $R-r$ in each case.

What must be the position of two circles, then, if the distance between their centers is

i. 0?

ii. Less than the difference of their radii?

iii. Equal to the difference of their radii?

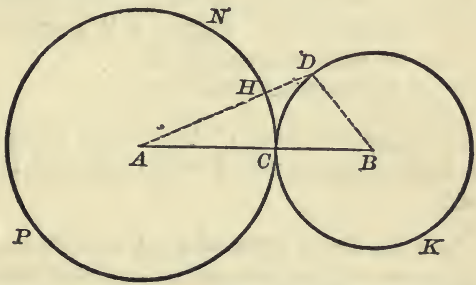
iv. Less than the sum, and greater than the difference, of their radii?

v. Equal to the sum of their radii?

vi. Greater than the sum of their radii?

323. If two circles be tangent to each other externally, the point of contact and the center of the two circles are collinear.

Post. Let the two circles PCN and CDK be tangent to each other externally, the point of contact being C, and let A and B be their respective centers.



To Prove. A, C, and B are in the same straight line, *i.e.* are collinear.

Cons. Let the radii AC and BC be drawn, and BD any other radius of the circle DCK, and points A and D joined.

If we can prove that ACB is the shortest line between A and B, then it is a straight line, and the three points are collinear.

So we are to prove that ACB is the shortest distance from A to B.

Dem. Since every point in the circumference CDK except C is outside the circumference CPN (see Def.), AD must extend beyond the circumference.

$AD > AH$. Why?

$AC = AH$. Why?

$\therefore AD > AC$. Why?

$DB = CB$. Why?

$\therefore AD + DB > AC + CB$. Why?

But D is any point in the circumference CDK except C. Hence, the distance from A to B by way of D is always greater than by way of C. Or ACB is the shortest distance between A and B, and is therefore a straight line.

$\therefore$ A, B, and C are collinear since they are in the same straight line. Q.E.D.

324. If two circles are tangent to each other internally, their centers and point of contact are collinear.

325. If the circumferences of two circles intersect each other, the line joining their centers will bisect their common chord and be perpendicular to it.

Sug. Consult 103 or 95 and 141.

326. If the circumferences of two equal circles intersect each other, their common chord will bisect the line joining their centers.

327. If from the same point without a circle two tangents be drawn, these two tangents are equal; *i.e.* the distances from the common point to the points of contact are equal.

Sug. Consult 134.

328. If from a point in a secant that passes through the center of the circle two tangents be drawn, the secant will bisect the angle formed by the two tangents, the chord joining the points of contact, and its subtended arc.

MEASUREMENT, RATIO, AND PROPORTION.

329. To *measure* a quantity of any kind is to find how many times it contains another quantity of the same kind used as the *unit;* thus, to measure a line is to find the number which expresses how many times it contains another line called the *unit of length,* or *linear unit,* as the *foot, yard, rod,* etc. This number is called the *numerical measure* of that quantity.

330. If of two unequal magnitudes the less is contained an exact number of times in the greater, the latter is said to be a *multiple* of the former, and the former an *aliquot part* of the latter.

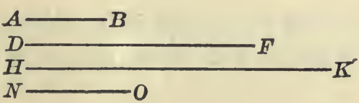
331. If of three unequal magnitudes the smallest is contained an exact number of times in each of the two larger, the latter are said to be *commensurable,* and the former is said to be their *common measure.*

332. When no magnitude can be found which is contained an exact number of times in each of the two larger, the latter are said to be *incommensurable.*

333. If two commensurable magnitudes contain their common measure, one n times and the other m times, they are said to be to each other as n to m, or in the ratio of n to m.

For example, if the line AB is contained in the line DF 3 times, and in line NR 5 times, then DF is to NR as 3 to 5, and the line AB is a common measure of the two lines DF and NR.

334. *Equimultiples* of two or more magnitudes are the results obtained by multiplying these magnitudes by the same number (see 46).



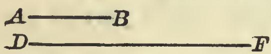
Thus, if AB and NO be any two lines, and DF and HK two other lines such that

the former contains the line AB n times (in this particular case n is 3), and the latter contains the line NO n times, then DF and HK are *equimultiples* of AB and NO respectively.

335. If two magnitudes are to each other as m to n (see 333), it is usually expressed thus, $m:n$, called the *ratio* form; or thus, $\frac{m}{n}$, called the *fractional* form.

336. A Ratio may be defined as an expression of comparison, in respect to size, of two magnitudes of the same kind.

337. If AB and DF are two magnitudes, and DF a multiple of AB (see 330), the latter being contained in the former n times, then the ratio of DF to AB is as n to unity, or $n:1$.



338. If x and y are commensurable (331), and their common measure is contained n times in x, and m times in y, then the ratio of x to y is as n to m, or, technically expressed, $n:m$.

339. If x and y are *incommensurable* (see 332), the ratio cannot be exactly expressed. If, however, x and y are numbers, the ratio may be *approximately* expressed by placing 1 for the first term, and the quotient of the greater divided by the less to any number of decimal places for the second term.

340. The first term of a ratio is called the *antecedent*, and the second term the *consequent*.

341. A Proportion is an expression of equality between two equal ratios, usually indicated by four dots between the two ratios; thus, $a:b::x:y$, and read, a is to b as x is to y, or $\frac{a}{b}=\frac{x}{y}$ called the *equation* form.

342. The four magnitudes forming a proportion are called *proportionals*.

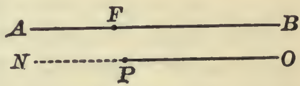
343. If a proportion contains only two ratios, it is called a *simple* proportion; if more than two and all equal, it is called a *continued* proportion.

344. The first and last terms of a simple proportion are called the *extremes*, and the other two the *means.*

345. In a simple proportion, where the terms are all of different values, each term is said to be a *fourth proportional* to the other three.

346. In a simple proportion, where the two means or the two extremes are alike, the repeated quantity is said to be a *mean proportional* to the other two, and the proportion is called a *mean proportion;* also, either of the quantities not repeated is said to be a *third proportional* to the other two.

347. When a line is divided by a point between its extremities, it is said to be divided into segments *internally;* when the point of division is on the extension of the line, it is said to be divided *externally.* Thus, AB is divided internally at F, and PO is divided externally at N. The segments in both cases are the distances between the point of division and the ends of the line.



348. When a line is so divided that the larger part is a mean proportional between the whole line and the smaller part, it is said to be divided in *extreme and mean ratio;* and when it is divided both internally and externally into segments having the same ratio, it is said to be divided *harmonically.*

349. It will be found on investigation that the *order* of four proportionals, *when all of the same kind,* can be varied, as well as certain other transformations effected, without destroying the proportion. The principal of these are as follows:

350. Magnitudes are said to be in proportion by *alternation* when either the two means or the two extremes are made to exchange places.

351. Magnitudes are said to be in proportion by *inversion* when the means are made to exchange places with the extremes.

352. Magnitudes are said to be in proportion by *composition* when the sum of the terms of the first ratio is to either term of the first ratio as the sum of the terms of the second ratio is to the *corresponding* term of that ratio.

353. Magnitudes are said to be in proportion by *division* when the difference of the terms of the first ratio is to either term of that ratio as the like difference of the terms of the second ratio is to the *corresponding* term of that ratio.

354. Magnitudes are said to be in proportion by *composition and division* when the sum of the terms of the first ratio is to their difference as the sum of the terms of the last ratio is to their *like* difference.

THEOREMS.

355. In the following theorems the "product of two quantities" or the "product of two magnitudes" means the product of the numbers representing their numerical measurement. It is assumed that pupils are familiar with elementary algebraic processes.

356. If four magnitudes are proportionals, the product of the means equals the product of the extremes.

Given $a : x :: n : r$.

To Prove $ar = nx$.

Dem. $a : x :: n : r$. (By Hyp.)

$\therefore \frac{a}{x} = \frac{n}{r}$. (Changing to equation form.)

$\therefore a = \frac{nx}{r}$. (Ax. viii.)

$\therefore ar = nx$. (Ax. viii.)

Q.E.D.

Sch. The fact stated in the above theorem is the test of a proportion, and may be applied to settle any doubt or question regarding it.

357. **If the product of two quantities equals the product of two others, the factors of either product may be made the means, and the other two the extremes, of a proportion.**

Post. Let $cx = an$.

We are required to form a proportion from the four factors c, x, a, and n, in which c and x shall be the extremes.

Dem. $\qquad cx = an. \qquad \therefore \dfrac{c}{a} = \dfrac{n}{x}.$

$\qquad\qquad (?) \qquad\qquad (?)$

$\qquad \therefore c = \dfrac{an}{x}. \qquad \therefore c : a :: n : x.$

$\qquad\qquad (?) \qquad\qquad (?)$

Q.E.D.

Take the same equation and form other proportions, *i.e.* make the terms appear in a different order.

Form proportions from the following equations in which the factors of the product marked *Ex.* shall form the extremes, avoiding the factor 1.

i. $\overset{\text{Ex.}}{2x} = ac.$

ii. $6 = \overset{\text{Ex.}}{ax}.$

iii. $\overset{\text{Ex.}}{3\sqrt{a}} = 14.$

iv. $a(x + y) = \overset{\text{Ex.}}{n^2}.$

v. $y^2 = \overset{\text{Ex.}}{an + ac}.$

vi. $\overset{\text{Ex.}}{ab + bx} = cn + yc.$

vii. $ax + x = \overset{\text{Ex.}}{yb^2 + b^2}.$

viii. $\overset{\text{Ex.}}{x^2 - 1} = a^2 - b^2.$

ix. $\overset{\text{Ex.}}{ax + xb + cx} = nd + hn + nk.$

x. $x^2 + 2cx + c^2 = \overset{\text{Ex.}}{an + ny}.$

358. If four quantities form a proportion, they will be in proportion by ***Inversion.*** *Sug.* Consult 357.

359. If four quantities form a proportion, they will be in proportion by ***Alternation.*** *Sug.* Consult 357.

360. If four quantities form a proportion, they will be in proportion by ***Composition.***

Post. Let the four quantities x, y, a, and b form a proportion, so that

$$x : y :: a : b.$$

To Prove. i. $x + y : y :: a + b : b.$

ii. $x + y : x :: a + b : a.$

Dem. $x : y :: a : b.$ (By Hyp.)

$\therefore \frac{x}{y} = \frac{a}{b}.$ Why?

$\frac{x}{y} + 1 = \frac{a}{b} + 1.$ Why?

$\frac{x}{y} + \frac{y}{y} = \frac{a}{b} + \frac{b}{b}.$ Why?

$\frac{x+y}{y} = \frac{a+b}{b}.$ Why?

$\therefore x + y : y :: a + b : b.$ Why? Q.E.D.

The pupil should demonstrate Part ii.

361. If four quantities form a proportion, they will be in proportion by ***Division.***

Sug. There are four cases. In Case i. subtract 1 from each member of the equation. The other cases may be similarly demonstrated by consulting 358.

362. If two proportions have a ratio in each equal, the other two ratios will form a proportion.

363. If two proportions have the two antecedents of one equal respectively to the two antecedents of the other, the consequents will form a proportion.

364. If two proportions have the two consequents of one equal respectively to the two consequents of the other, the antecedents will form a proportion.

365. If the terms of one ratio of a proportion are equal, the terms of the other ratio or ratios are also equal.

366. If four quantities form a proportion, they will be in proportion by ***Composition and Division.***

Sug. Consult 360 and 361, and 363 or 364.

367. If any number of magnitudes of the same kind form a proportion, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.

Post. Let the quantities a, x, c, n, d, and r form a continued proportion, so that

$$a : x :: c : n :: d : r.$$

To Prove. i. $a + c + d : x + n + r :: a : x$;

or ii. $:: c : n$;

or iii. $:: d : r$.

Dem. $a : x :: c : n :: d : r$. (By Hyp.)

$\therefore$ (1) $ax = ax$; Why?

(2) $cx = an$; Why?

(3) $dx = ar$. Why?

Hence, $ax + cx + dx = ax + an + ar$, Why?

or $x(a + c + d) = a(x + n + r)$. Why?

The pupil should be able to finish this case, and also demonstrate the other two without difficulty. Consult 357.

368. If four quantities form a proportion, the terms of either ratio may be either multiplied or divided by the same quantity, and the results still form a proportion.

369. If the antecedents or consequents of a proportion be either multiplied or divided by the same quantity, the results will still form a proportion.

370. If four quantities form a proportion, and the terms of one ratio be either multiplied or divided by the same quantity, while both terms of the other ratio be either multiplied or divided, either by the same or different quantity from that used in the first ratio, the results will still form a proportion.

Sug. The pupil should take each case involved in the statement of the above theorem separately.

371. If four magnitudes form a proportion, and the antecedents be either multiplied or divided by the same quantity, while the consequents are both multiplied or divided by another quantity, the results will form a proportion.

372. **If two proportions be given, the products of the corresponding terms will also form a proportion.**

373. If two proportions be given in which two corresponding ratios have the antecedent of one equal to the consequent of the other, the remaining antecedent and consequent, together with the products of the corresponding terms of the other two ratios, will form a proportion.

374. If the antecedents of a proportion are equal, the consequents are equal.

375. Converse of 374.

376. If four quantities form a proportion, the reciprocals of the terms of either ratio are inversely proportional to the terms of the other ratio.

377. **If four quantities form a proportion, their like powers and like roots will also form a proportion.**

378. If three quantities form a proportion, the first is to the third as the square of the first is to the square of the second; *i.e.* if $a : n :: n : x \therefore a : x :: a^2 : n^2$.

379. **If three terms of a simple proportion are equal respectively to the three corresponding terms of another proportion, the fourth terms of the two proportions are equal.**

380. **Equimultiples of two magnitudes are in the same ratio as the magnitudes themselves.**

THEORY OF LIMITS.

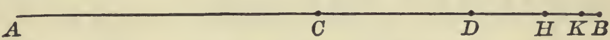
381. **A Constant Quantity,** or simply a *Constant,* is a quantity whose value remains unchanged throughout the same discussion.

382. **A Variable Quantity,** or simply a *Variable,* is a quantity which may assume different values in the same discussion, according to the conditions imposed.

383. **The Limit** of a variable is a constant quantity, which the variable is said to *approach* in value whenever a *regular* and *definite increase* or *decrease* in value is assigned to the latter.

384. Whenever it can be shown that the value of a *variable,* by such constant increase or decrease in value, can be made to differ from that of its limit by less than any appreciable or assignable quantity, however small, this variable is said to *approach indefinitely to its limit.*

For example:



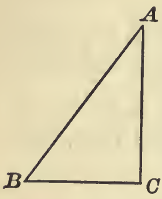
Suppose a point move from A toward B under the condition that during the first second it shall move over one half the distance AB, or AC, and that during each successive second it shall move over one half the remaining distance. Then at the end of the second second it would be at D, at the end of the third at H, at the end of the fourth at K, and so on. It is evident that it can never reach the point B, for there will constantly remain *one half* the distance; but if its motion be continued indefinitely, it will approach indefinitely near to B.

Consequently, the distance from A to the moving point is an *increasing variable* and AB is its limit; while the distance from B to the moving point is a *decreasing variable,* with zero as its limit.

Other illustrations may be given, *e.g.:*

$$0.3333 + \cdots = \tfrac{3}{10} + \tfrac{3}{100} + \tfrac{3}{1000} + \tfrac{3}{10000} + \cdots.$$

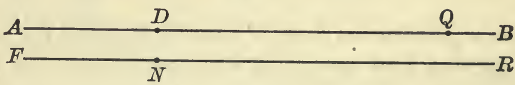
Here the sum of the series of fractions is the increasing variable, and approaches $\frac{1}{3}$ as its limit.



Again: let ABC be a right triangle, with C the right angle, and consider the point B to move toward C; the angle A will then be a *decreasing variable* approaching zero as its limit, and the angle B will be an *increasing* variable approaching a right angle as its limit.

385. It will be evident to the pupil that if the numerator of a fraction be a variable and the denominator a constant, then the entire fraction may be regarded as a variable, increasing or decreasing as the numerator is an increasing or decreasing variable. Similarly, if the numerator be a constant and the denominator a variable, the entire fraction is a variable, increasing or decreasing according as the denominator is a decreasing or increasing variable.

386. *Theorem.* **If two variables are always equal, and each approaches a limit, their limits are equal.**



Post. Let AB and FR be the limits to which the two equal variables AD and FN indefinitely approach.

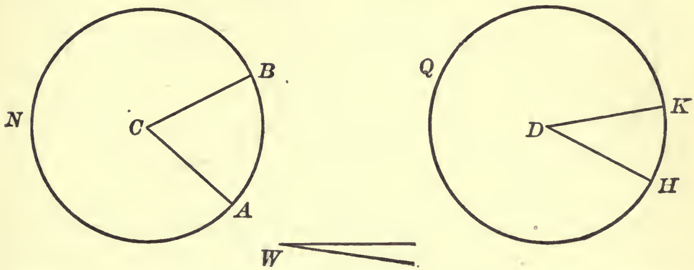
To Prove. $AB = FR$.

Dem. AB and FR are either equal or unequal. Let us suppose them unequal, and that AB is the greater. Mark off AQ equal to FR.

Then the variable FN cannot exceed FR, but the variable AD may exceed AQ, and consequently the variable AD becomes greater than the variable FN. This, however, is contrary to the hypothesis that the two variables must always be equal. Therefore, AB and FR cannot be unequal; *i.e.* they are equal.

Q.E.D.

387. **In the same or equal circles two central angles are in the same ratio as the arcs which their sides intercept.**



Post. Let NBA and QKH be two equal circles, and C and D two central angles.

To Prove. $\angle C : \angle D : :$ arc AB : arc HK.

CASE I. When the angles are *commensurable.*

Dem. If the angles are commensurable, there is some angle, as W, which will be contained an exact number of times in each.

Suppose it is contained m times in $\angle C$ and n times in $\angle D$.

Then $\angle C : \angle D : : m : n$.

If, now, lines be drawn from C and D dividing the two angles into m and n, equal parts respectively, each part being equal to angle W, then arc AB will be divided into m equal arcs, and HK into n equal arcs, the divisions *all* being equal, by Theorem 291.

$\therefore$ arc AB : arc $HK : : m : n$.

$\therefore \angle C : \angle D : :$ arc AB : arc HK.

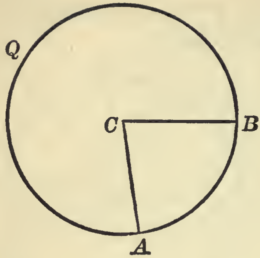
(?)

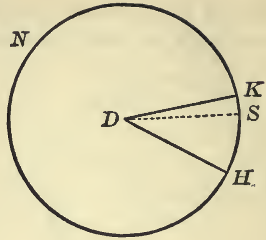
CASE II. When the angles are *incommensurable.*

Post. Let AQB and NHK be two equal circles, and C and D two central angles which are *incommensurable.*

To Prove. $$\frac{\angle D}{\angle C} = \frac{\text{arc } HK}{\text{arc } AB},$$

or $\angle D : \angle C : :$ arc HK : arc AB.





Dem. Conceive the $\angle C$ to be divided into any number of equal parts, and one of these parts to be applied as a unit of measure to the $\angle D$. It will be contained a certain number of times with a remainder, SDK, less than the unit of measure.

If, now, the number of equal parts into which the $\angle C$ is divided be indefinitely increased, the unit of measure of the $\angle HDK$ will be correspondingly diminished, and the point S will get indefinitely near to K.

$\therefore$ $\angle HDS$ becomes an incr. var. appr. its limit $\angle HDK$.

$\therefore$ $\dfrac{\angle HDS}{\angle C}$ becomes an incr. var. appr. its limit $\dfrac{\angle HDK}{\angle C}$.

In like manner,

arc HS becomes an incr. var. appr. its limit arc HK.

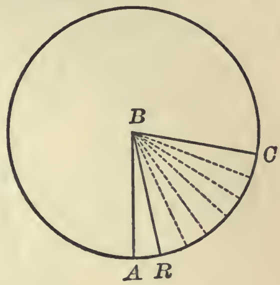
$\therefore$ $\dfrac{\text{arc } HS}{\text{arc } AB}$ becomes an incr. var. appr. its limit $\dfrac{\text{arc } HK}{\text{arc } AB}$.

But $$\frac{\angle HDS}{\angle C} = \frac{\text{arc } HS}{\text{arc } AB} \quad \ldots\ldots \quad \text{Case I.}$$

$$\therefore \frac{\angle HDK}{\angle C} = \frac{\text{arc } HK}{\text{arc } AB} \quad \ldots\ldots \quad (386)$$

or $$\angle D : \angle C :: \text{arc } HK : \text{arc } AB. \qquad \text{Q.E.D.}$$

388. If in the same or equal circles one central angle be used as a unit to measure another, and its intercepted arc, as a unit to measure the other arc, then their numerical measures are equal.



Post. Let the central angle ABR be used as a unit to measure the central angle ABC, and arc AR as a unit to measure the arc AC.

To Prove. That their numerical measures are equal.

Dem. $\angle ABC : \angle ABR :: \text{arc } AC : \text{arc } AR$ (Theorem 387)

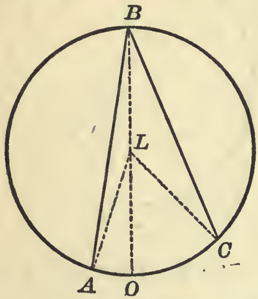
or
$$\frac{\angle ABC}{\angle ABR} = \frac{\text{arc } AC}{\text{arc } AR}.$$

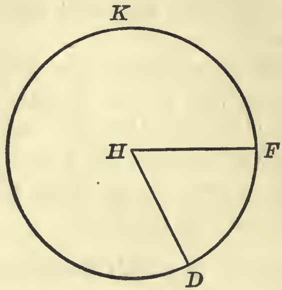
That is, the angle ABR is contained in the angle ABC the same number of times that the arc AR is contained in the arc AC.

Hence their numerical measures are equal. Q.E.D.

389. *Sch.* If the degree (see 185) were used to measure the angle, and its corresponding arc to measure the arc, then, from the above demonstration, there would be the same number of angle degrees as arc degrees; and so the arc may be said *to measure the angle.* Formal expression to this idea is commonly given as follows, viz.: *A central angle is measured by the arc which its sides intercept on the circumference.* Hence, whatever numerical ratio exists between two arcs in the same or equal circles, the same numerical ratio exists between the central angles whose sides intercept those arcs; *e.g.* if in the same or equal circles two central angles intercept arcs one of which is twice as large as the other, then one angle is twice as large as the other, etc. Similarly, if an angle is measured by one fourth the circumference, this angle is a right angle; and if two angles are together measured by one half the circumference, these angles are supplemental, etc.

390. If in the same or equal circles, an inscribed angle and a central angle intercept the same or equal arcs, the central angle is double the inscribed angle.





Sug. Consult 292 and 129.

391. An inscribed angle is measured by one half the arc intercepted by its sides. In demonstrating this theorem it will simplify it somewhat to make three cases of it; viz:

I. When one of the sides of the angle is a diameter.

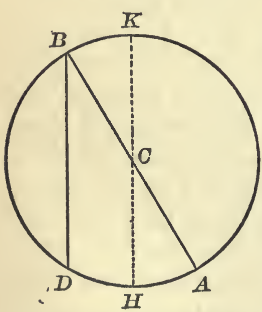
II. When the center is between the sides of the angle.

III. When the center is without the angle.

Post.

To Prove.

Cons. Draw a diameter, as HK, parallel to BD.



Dem. What relation exists between the angles BCK and HCA? Why?

What relation, then, exists between the two arcs HA and BK? Why?

What relation between the two arcs DH and BK? Why?

What relation, then, between the two arcs DH and AH? Why?

What relation, then, between the two arcs AH and AD? Why?

What relation between the two angles HCA and B? Why?

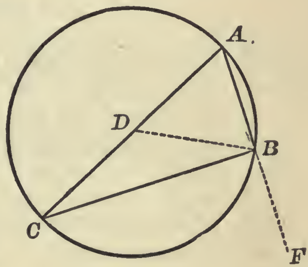
What measures the angle HCA? Why?

What, then, must measure the angle B? Why?

After answering correctly the foregoing questions the pupil should have no difficulty in writing out a complete demonstration of each of the three cases, using Case I in demonstrating II, etc.

392. If two angles be inscribed in the same or equal circles, and their sides intercept the same or equal arcs, the two angles are equal.

393. Converse of 392.



394. An angle inscribed in a semicircle is a right angle.

Sug. Prove $\angle ABC = \angle CBF$.

395. An angle inscribed in a segment greater than a semicircle is an acute angle, and an angle inscribed in a segment smaller than a semicircle is an obtuse angle.

396. If an angle be formed by two chords whose vertex is between the center and circumference, it will be measured by one half the sum of the arcs intercepted by its sides and the sides of its vertical angle.

397. If the vertex of an angle formed by two secants is without the circle, this angle is measured by one half the difference of the two intercepted arcs.

398. If an angle be formed by a tangent and chord, this angle is measured by one half the arc intercepted by its sides.

399. If the vertex of an angle formed by a tangent and secant is without the circle, this angle is measured by one half the difference of the two intercepted arcs.

400. The angle formed by two tangents to the same circle is measured by one half the difference of the two intercepted arcs.

401. The opposite angles of an inscribed quadrilateral are supplemental.

402. *Sch.* The principle enunciated in 389 with reference to central angles may now be stated with reference to *any two angles* relating to the same or equal circles; viz:

If, in the same or equal circles, two angles are known to be in a certain numerical ratio, their measuring arcs will be in the same ratio, and conversely.

ADVANCE THEOREMS.

403. If, from the same point, two tangents to a circle be drawn whose points of contact are the extremities of a chord, the angle formed by the two tangents is double the angle formed by the chord and diameter drawn from either extremity of the chord.

404. The bisectors of the angles of a circumscribed quadrilateral are concurrent.

405. If two circles which are tangent to each other externally have three common tangents, the one that passes through the point of contact of the two circles bisects the other two.

Sug. Consult 327.

406. If four tangents form a quadrilateral, the sum of two opposite sides equals the sum of the other two opposite sides.

407. If two mutually equiangular triangles be inscribed in equal circles, the triangles are equal in every respect.

408. If two opposite sides of an inscribed quadrilateral are equal, the other two sides are parallel.

409. Converse of 408.

410. If the opposite angles of a quadrilateral are supplemental, it is inscriptible.

411. If two equal chords be drawn from opposite extremities of a diameter and on opposite sides of it, they will be parallel.

412. If two chords be drawn through the same point in a diameter making equal angles with it, they are equal.

413. If one of the equal sides of an isosceles triangle be the diameter of a circle, the circumference will bisect the base.

414. If an equilateral triangle be inscribed in a circle, and a diameter be drawn from one vertex, the triangle, formed by joining the other extremity of the diameter and the center of the circle with one of the other vertices of the inscribed triangle, will also be equilateral.

415. The bisectors of the angles formed by extending the sides of an inscribed quadrilateral are perpendicular to each other.

416. If an equilateral triangle be inscribed in a circle, and any point in the circumference be selected at random, one of the lines which join this point to the three vertices will equal the sum of the other two.

417. If an inscribed isosceles triangle have its base angles each double the vertical angle, and its vertices be the points of contact of three tangents, these tangents will form an isosceles triangle each of whose base angles is one third its vertical angle.

418. If through any point selected at random in a circle a diameter and other chords be drawn, the least chord will be the one that is perpendicular to the diameter.

419. If the sides of any quadrilateral be the diameters of circles, the common chord of any adjacent two is parallel to the common chord of the other two.

420. If equilateral triangles be described on the sides of any triangle, the circumferences of their circumscribed circles are concurrent.

421. If equilateral triangles be described on the sides of any triangle, the lines joining the centers of these triangles will form an equilateral triangle.

Sug. Draw the circumscribed circles.

PROPORTIONAL LINES.

422. If a series of parallel transversals, intersecting any two straight lines, intercept equal distances on one of these lines, they will also intercept equal distances on the other.

Sug. Make such construction as will enable you to use theorems 203 and 219.

423. If a line be drawn parallel to one side of a triangle, the four parts into which it divides the other sides will form a proportion.

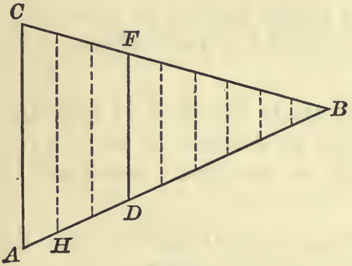


FIG. I.

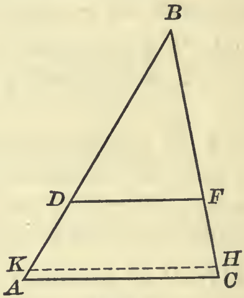


FIG. II.

Post. Let ABC be a triangle with line DF drawn parallel to AC.

To Prove. AD, DB, CF, and BF are proportionals.

CASE I. When AD and DB, Fig. I, are commensurable.

Dem. Since AD and DB are commensurable, they must have a common measure, as AH.

Then AH will be contained in AD m times and in DB n times.

$$\therefore AD : DB :: m : n. \quad (?)$$

Now, at the several points of division on AB suppose lines are drawn to CB parallel to DF.

What will be the relation of these lines to one another? Why?

These lines will divide CB into equal parts. Why?

CF will be divided into m equal parts and FB into n equal parts.

$$\therefore CF : FB :: m : n. \quad (?)$$

$$\therefore AD : DB :: CF : FB. \quad (?)$$

CASE II. When AD and DB, Fig. II, are incommensurable.

Dem. Suppose BD to be divided into any number of equal parts and one of these taken as a unit to measure AD. This will be contained a certain number of times with a remainder AK less than the unit.

Now let KH be drawn parallel to DF.

$$\therefore KD : DB :: FH : BF \quad \text{(By Case I).}$$

or
$$\frac{KD}{DB} = \frac{FH}{BF}.$$

Suppose now the unit of measure to be indefinitely diminished.

$\therefore$ KA would become a decreasing variable and DK would become an increasing variable approaching AD as its limit.

Likewise FH would also become an increasing variable with FC as its limit.

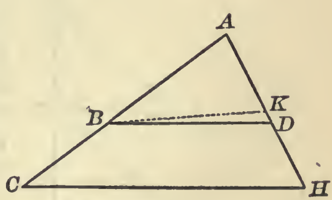
Consequently $\frac{KD}{DB}$ and $\frac{FH}{BF}$, since their denominators are constants, become increasing variables approaching their respective limits $\frac{AD}{DB}$ and $\frac{FC}{BF}$.

But
$$\frac{KD}{DB} = \frac{FH}{BF}.$$

$$\therefore \frac{AD}{DB} = \frac{FC}{BF}. \quad (?)$$

$\therefore$ $AD : DB :: FC : BF$. (Proportion form.) Q.E.D.

424. If a line divide two sides of a triangle into proportional parts, this line is parallel to the third side.



Post. Let ACH be a triangle, with line BD drawn so that

$$AB : BC :: AD : DH.$$

To Prove. BD is parallel to CH.

Cons. Suppose BK to be drawn through B parallel to CH.

Dem. $AB : BC :: AD : DH$. Why?

$AB : BC :: AK : KH$. Why?

$\therefore AD : DH :: AK : KH$. Why?

$\therefore AD + DH : DH :: AK + KH : KH$. Why?

$AH : DH :: AH : KH$. Why?

$\therefore DH = KH$. Why?

$\therefore$ points D and K must coincide Why?

What, then, must be the relative position of the lines DK and BD? Why?

But BK was drawn parallel to CH. Consequently BD, which coincides with BK, must be parallel to CH. Q.E.D.

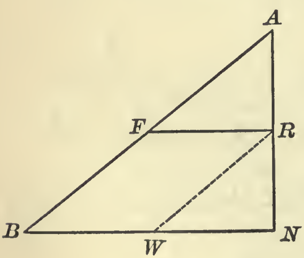
425. If a line be drawn parallel to one side of a triangle, the other two sides and either pair of corresponding parts are proportionals.

Sug. Consult 423 and 360.

426. If a line meet two sides of a triangle so that those two sides and either pair of corresponding parts are proportionals, this line is parallel to the third side.

Sug. Proceed as in 424.

427. If two lines be drawn parallel to one side of a triangle intersecting the other two sides, the parts thus intercepted are proportionals.



428. If a line be drawn parallel to one side of a triangle, this side, its parallel, and either of the other two sides, together with the part of the latter joining the third side, are proportionals.

To Prove. $BN : FR :: AN : AR$.

Cons. Draw $RW \parallel$ to AB.

429. If a line be drawn parallel to the base of a triangle, and another from vertex to base, the parts of both the base and its parallel are proportionals.

430. The bisector of an angle of a triangle divides the opposite side into segments proportional to the other two sides.

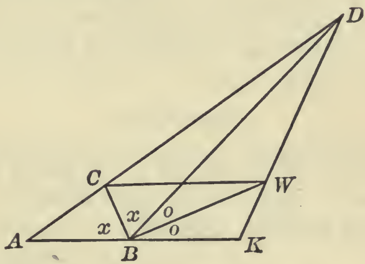
Sug. From the vertex of the bisected angle extend one of the sides, and from one of the other vertices draw a line parallel to the bisector meeting the side extended.

431. If the bisector of an exterior angle of a triangle meet one of the sides extended, it will divide this side externally into segments proportional to the other two sides of the triangle.

Sug. From extremity of the side extended, draw a parallel to the bisector.

432. If the median of a triangle be drawn and the angles which the median makes with the base be bisected, the line joining the extremities of the bisectors will be parallel to the base.

Sug. Consult 430 and 426.



SIMILAR FIGURES.

433. Similarity in general means *likeness of form; i.e.* two figures are said to be similar when they have the same shape, although they may differ in size.

Similar geometrical figures are those whose homologous angles are equal, and whose homologous sides form a proportion.

434. *Homologous sides* of similar figures are those which join the vertices of equal angles. In similar triangles the homologous sides are those that are *opposite* the equal angles. The pupil should be careful to observe at the outset that similarity involves two things, viz. *equality of angles* and *proportionality of sides.*

435. Two geometrical figures are said to be *mutually equiangular* when the angles of one are equal respectively to the corresponding angles of the other, and *mutually equilateral* when each side of one has an equal side in the other.

436. Similar *arcs, sectors,* and *segments,* are those that correspond to equal central angles.

It is obvious that if two geometrical figures are similar to the same or equal figures, these two are similar to each other.

THEOREMS.

437. If a line be drawn parallel to one side of a triangle, the triangle thus formed is similar to the given triangle.

Sug. Consult 434, 425, and 428.

438. If two triangles are mutually equiangular, they are similar.

Sug. Consult 437.

439. If two angles of one triangle are equal respectively to two angles of another, the two triangles are similar.

440. If two right triangles have an acute angle in each equal, the two triangles are similar.

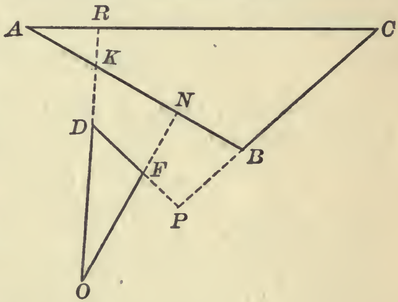
441. If two triangles have an angle in each equal, and the sides including those angles form a proportion, the two triangles are similar.

Sug. Consult 437.

442. If two triangles have their sides respectively parallel, they are similar.

Sug. Produce the non-parallel sides until they intersect, then consult 114.

443. If two triangles have their sides respectively perpendicular to each other, they are similar.



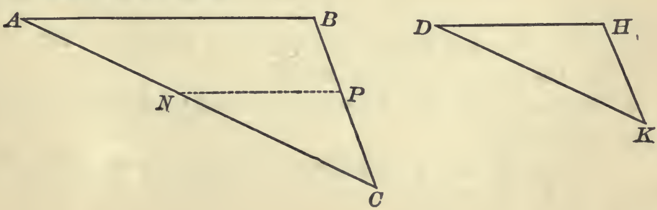
Sug. Extend the ⊥ sides until they meet, then consider the angles of the quadrilateral *NFPB* and the angles at *B*. Then compare the ∡ *A* and *O* by means of the △ *ARK* and *ONK*.

444. If two isosceles triangles have equal vertical angles, they are similar.

445. If two triangles be similar, their homologous altitudes are in the same ratio as either pair of homologous sides which include the vertical angle.

446. If two triangles be similar, their homologous altitudes are in the same ratio as their homologous bases.

447. If the homologous sides of two triangles form a proportion, the triangles are similar.



Post. Let ABC and DHK be two triangles, their homologous sides forming the continued proportion

$$AB : DH :: AC : DK :: BC : HK.$$

To Prove. That the two triangles are similar; *i.e.* that they are mutually equiangular.

Cons. Make NC equal to DK, PC equal to HK, and join NP.

Dem. In the given proportion substitute for DK and HK their equals NC and PC. Then

$$AC : NC :: BC : PC.$$

What is true, then, of the two triangles ABC and NPC? See Theorem 441.

$$\therefore\ AB : NP :: BC : PC. \quad \text{Why?}$$

But by Hyp., $\quad AB : DH :: BC : KH.$

What is true of the first terms of these two proportions? Of the third terms? Of the last terms? What must be true, then, of the second terms?

The pupil should finish the demonstration without difficulty.

448. If, in a right triangle, a perpendicular be drawn to the hypothenuse from the vertex of the right angle, —

i. The two triangles thus formed are similiar to the original triangle and to each other.

ii. The perpendicular and the two segments of the hypothenuse are proportionals.

iii. Either leg, the hypothenuse, and that segment of the latter which joins the former, are proportionals.

iv. The two segments of the hypothenuse and the squares of the two legs are proportionals.

Sug. Use the two proportions of iii. by Theorem 356, and divide one by the other.

v. The square of the hypothenuse bears the same ratio to the square on either leg as the hypothenuse bears to that segment of the latter joining that leg.

Sug. Consult Part iii. and 378.

vi. The two legs, the hypothenuse, and the perpendicular are proportionals.

Sug. Select the two similar triangles and pick out the homologous sides.

449. If from any point, in the circumference of a circle, a perpendicular be drawn to a diameter, this perpendicular will be a mean proportional between the two segments of the diameter.

Sug. Consult 394.

450. If two chords of a circle intersect each other, the four parts are proportionals.

Sug. Draw other chords and compare the triangles.

451. If two secants be drawn from the same point without the circle, the entire secants and their external segments are proportionals.

Sug. Draw such chords as will make similar triangles.

452. If, from the same point, a tangent and secant to a circle be drawn, the tangent, the secant, and the external segment of the latter, are proportionals.

453. If two circles intersect each other, the two tangents from any point in their common secant are equal.

454. If two circles intersect each other, their common secant bisects their common tangents.

POLYGONS.

455. **A Polygon** is a portion of a plane bounded by straight lines.

What is the least number of sides that a polygon can have? The greatest number?

456. The word *polygon* is from two Greek words meaning *many angles.*

What is a polygon of three sides usually called? It is also called a *trigon.*

A polygon of four sides is usually termed what? It is also called a *tetragon.*

A polygon of five sides is called a *pentagon,* one of six sides a *hexagon,* one of seven sides a *heptagon,* one of eight sides an *octagon,* one of nine sides a *nonagon* or *enneagon,* one of ten sides a *decagon,* one of eleven sides an *undecagon,* one of twelve sides a *dodecagon,* and one of fifteen sides a *pentedecagon.*

457. **A Convex Polygon** is one each of whose angles is less than 180°, and

A Concave Polygon is one that has one or more reflex angles.

Whenever polygons are mentioned in this work, convex polygons are meant unless otherwise specified.

The Diagonal of a polygon is a line joining any two vertices not consecutive.

458. **A Regular Polygon** is one that is both equilateral and equiangular.

459. A polygon is *inscribed in a circle* when the vertices of all its angles are on the circumference; *i.e.* when its sides are chords of that circle. The circle is said then to be circumscribed about the polygon, and the radius of this circle is also called the *radius of the polygon.*

460. A polygon is *circumscribed about the circle* when its sides are all tangents to the circle. The circle is then said to be *inscribed* in the polygon, and the radius of this circle is called the *apothem* of the polygon.

461. The Perimeter of a polygon is the sum of its sides. How many angles has each of the above-named polygons? How does the number of sides compare, in each case, with the number of angles? How many angles, then, has a polygon of n sides? How many diagonals can be drawn from a single vertex in each of the above-named polygons? Compare the number of diagonals in each case with the number of sides. If the polygon has n sides, how many diagonals can be drawn from a single vertex? How many triangles are formed by drawing diagonals from a single vertex in each of the above-named polygons? How does the number of triangles compare with the number of sides? If the polygon has n sides, how many triangles would be formed by diagonals similarly drawn?

Fill out the following table:

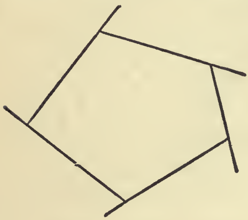
Diagram	No. of sides of the polygon	No. of diagonals drawn from one vertex	No. of △s formed	Value of ∠s in rt. ∠s
	3			
	4			
	5			
	6			
	7			
	8			
	⋮			
	⋮			
	⋮			
	n			

462. The sum of the interior angles of a polygon of n sides is equal to twice as many right angles as the polygon has sides, less four right angles; *i.e.* $2n - 4$ right angles.

Sug. Let n equal the number of sides and consult the above table.

Cor. If a polygon of n sides is equiangular, each angle is equal to $\frac{2n-4}{n}$ rt. $\measuredangle$s.

463. The sum of the exterior angles of any polygon formed by extending each of its sides in succession similarly, is equal to four right angles.



Sug.

$$\begin{array}{rl} \text{Int. } \measuredangle\text{s} + \text{Ext. } \measuredangle\text{s} &= 2n. \\ \text{Int. } \measuredangle\text{s} &= 2n - 4. \\ \hline \therefore \text{ Ext. } \measuredangle\text{s} &= 4. \end{array}$$

Q.E.D.

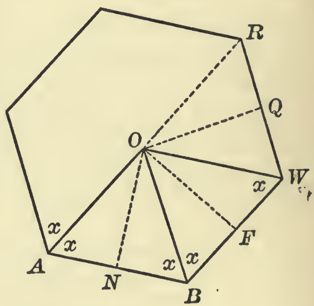
464. The bisectors of the angles of a regular polygon are concurrent.

Post. Let ABW, etc., be a regular polygon, and let AO and BO bisect $\measuredangle$s A and B, and let OW be drawn.

To Prove. OW bisects $\angle W$.

Cons. Draw $\perp$s ON, OF, and OQ.

Dem. Compare the $\triangle$s.



465. The perpendicular bisectors of the sides of a regular polygon are concurrent in the same point as the angle bisectors.

Sug. Draw angle bisectors and $\perp$s to the sides, then use above diagram.

466. If an equiangular polygon be circumscribed about a circle, the bisectors of its angles are concurrent at the center of the circle.

Sug. Consult 328.

467. If an equilateral polygon be inscribed in a circle, it is regular.

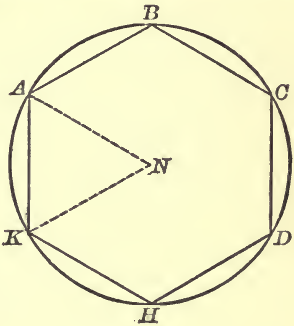
Sug. Consult 392.

468. If an equiangular polygon be circumscribed about a circle, it is regular.

Sug. Consult 466.

469. If a polygon be regular, a circle can be circumscribed about it.

470. If a polygon be regular, a circle can be inscribed in it.



471. One side of a regular hexagon is equal to the radius of the circumscribed circle.

Sug. Prove ANK an equilateral triangle by first comparing the angles.

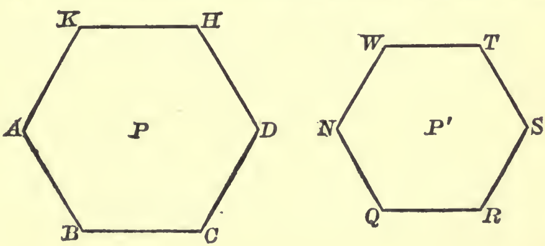
472. If the circumference of a circle be divided into any number of equal arcs, the chords joining the successive points of division will form a regular polygon.

473. If a regular polygon be circumscribed about a circle, the points of contact will divide the circumference into equal arcs.

474. If the circumference of a circle be divided into any number of equal arcs, the tangents at the points of division will form a regular polygon.

475. Tangents to a circumference at the vertices of a regular inscribed polygon form a regular circumscribed polygon of the same number of sides as the inscribed polygon.

476. Two regular polygons of the same number of sides are similar.



Post. Let P and P' be two regular polygons, each having n sides.

To Prove that they are similar; *i.e.* that their homologous sides form a proportion, and that they are mutually equiangular.

Sug. Compare AB and BC, etc., and NQ and QR etc., and thus obtain a proportion; then compare the $\angle$s A and N, B and Q, etc.

477. If two polygons be similar, the diagonals drawn from homologous vertices will divide them into the same number of trangles, similar two and two, and similarly placed.

Sug. Consult 441.

478. If the diagonals drawn from homologous vertices of two polygons divide them into the same number of triangles, similar two and two, and similarly placed, the two polygons are similar.

Sug. Consult 362.

479. The diagonals drawn from homologous vertices of two similar polygons are proportionals.

480. The perimeters of two similar polygons are in the same ratio as any two homologous sides and as any two homologous diagonals.

Sug. Consult 367.

481. The perimeters of two similar polygons are in the same ratio as the bisectors of any two homologous angles.

482. The perimeters of two regular polygons of the same number of sides are in the same ratio as the radii of their circumscribed circles.

Sug. Consult 444 and 480.

483. The perimeters of two regular polygons of the same number of sides are in the same ratio as their apothems.

PROBLEMS OF COMPUTATION.

484. Compute the value in right angles of the sum of the interior angles of a *pentagon, hexagon, octagon, decagon, dodecagon, pentedecagon,* and a polygon of *fifty-two* sides.

485. Compute the value, in terms of a right angle as the unit, of one of the angles of an *equiangular pentagon, octagon, dodecagon,* a polygon of *twenty* sides, *one hundred* sides.

486. Compute values of the above angles in degrees.

487. How many sides has an equiangular polygon, the sum of four angles of which is equal to seven right angles?

488. How many sides has an equiangular polygon, the sum of three angles of which is equal to five right angles?

489. How many sides has an equiangular polygon, the sum of nine angles of which is equal to sixteen right angles?

490. How many sides has the polygon, the sum of whose interior angles is equal to the sum of its exterior angles?

491. How many sides has the polygon, the sum of whose interior angles is double that of its exterior angles?

492. How many sides has the polygon, the sum of whose exterior angles is double that of its interior angles?

493. How many sides has the polygon, the sum of whose interior angles is equal to nine times the sum of its exterior angles?

494. How many sides has the equiangular polygon, when the sum of nine of its interior angles is four times the sum of its exterior angles?

495. How many sides has the equiangular polygon, when the sum of five of its interior angles is equal to two and one fourth times the sum of its exterior angles?

496. How many sides has the equiangular polygon, when the sum of four of it's interior angles equals seven and one half right angles?

497. How many sides has the equiangular polygon, when the sum of six of its interior angles equals three and three fourths times the sum of its exterior angles?

PROJECTION AND AREAS.

498. The Projection of one line upon another is that portion of the latter included between two perpendiculars to the latter drawn from the extremities of the former.

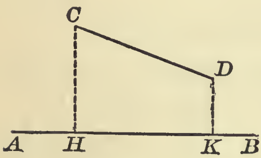


FIG. I.

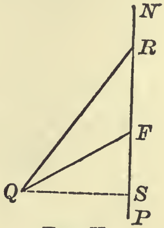


FIG. II.

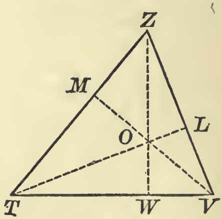


FIG. III.

For example, HK is the projection of the line CD upon AB, and RS is the projection of RQ on NP; it is also the projection of RQ on RF. Mention all the cases of projection in Fig. III, the dotted lines being perpendicular to the respective sides.

499. The Area of a surface is its numerical measure; *i.e.* the numerical expression for the number of times it contains another surface arbitrarily assumed as a unit of measure. For example, the area of the floor of a room is the number of times it contains some one of the common units of surface, square foot, square yard, etc.

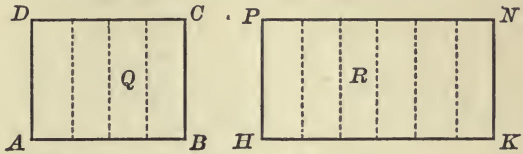
This unit of surface is called the *superficial unit.* The most convenient superficial unit is the square of the linear unit. The square of the linear unit (or line) is the surface of the square constructed with that linear unit for its sides.

Surfaces that are equal in area are said to be *equivalent.*

500. By the product of two lines is meant the product of the numbers which represent them when both are measured by the same linear unit. It is also sometimes expressed as the *rectangle of two lines.*

501. **If two rectangles have equal altitudes, their areas will be in the same ratio as their bases.**

CASE I. When the bases are commensurable.



Post. Let $ABCD$ and $HKNP$ be two rectangles, having their altitudes AD and HP equal, and their bases AB and HK commensurable. Designate their areas by Q and R respectively.

To Prove. $Q:R::AB:HK.$

Dem. Since AB and HK are commensurable, they must have a common measure. (331)

Suppose this common measure to be contained in AB n times, and in HK m times.

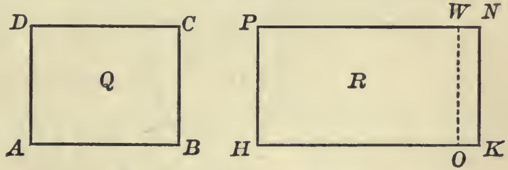
At the several points of division construct ⊥s to AB and HK. Then the rectangle $ABCD$ will be divided into n equal rectangles, and the rectangle $HKNP$ will be divided into m equal rectangles. These smaller rectangles are also equal to one another.

Hence $\qquad AB:KH::n:m, \qquad$ (?)

and $\qquad Q:R::n:m. \qquad$ (?)

$\therefore\ Q:R::AB:HK.$ (?)

CASE II. When the bases are incommensurable.



Dem. Suppose AB to be divided into any number of equal parts as n, and that one of these parts be applied to HK and found to be contained m times with a remainder less than that

part. From the points of division construct lines as in Case I. Then the rectangle $ABCD$ will be divided into n equal rectangles, and rectangle $NKHP$ into m equal rectangles, with a remaining rectangle less than one of the m rectangles.

Then, Rectangle Q : rectangle $PO :: AB : HO$. (By Case I.)

If, now, the unit of division be indefinitely diminished, base HO and Rect. PO become increasing variables approaching their respective limits KH and R.

$\therefore \dfrac{\text{Rect. } Q}{\text{Rect. } PO}$ becomes a decreasing variable approaching its limit $\dfrac{\text{Rect. } Q}{\text{Rect. } R}$, and $\dfrac{AB}{HO}$ becomes a decreasing variable approaching its limit $\dfrac{AB}{HK}$.

But $\qquad \dfrac{\text{Rect. } Q}{\text{Rect. } PO} = \dfrac{AB}{HO}. \qquad \therefore \dfrac{\text{Rect. } Q}{\text{Rect. } R} = \dfrac{AB}{HK}, \qquad$ (?)

or $\qquad$ Rect. Q : Rect. R $:: AB : HK.$ $\qquad$ Q.E.D.

502. **If two rectangles have equal bases, their areas are in the same ratio as their altitudes.**

Sug. Consider their altitudes as bases, and bases as altitudes, and proceed as in 501.

503. **The areas of any two rectangles have the same ratio as the products of their respective bases and altitudes.**

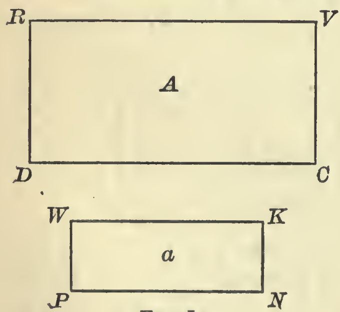


Fig. I.

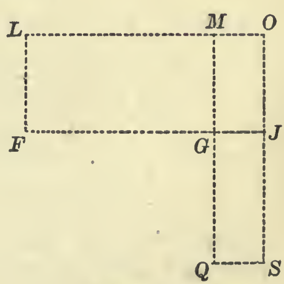


Fig. II.

Post. Let $RVCD$ and $WKNP$ be any two rectangles with bases DC and PN. Designate their areas by A and a respectively, their altitudes by H and h, and their bases by B and b.

To Prove. $A:a::B \cdot H:b \cdot h.$

Cons. Draw $FG=DC$ and prolong it, making $GJ=KN$. Through G draw $MQ \perp$ to FJ, making $MG=RD$, and $GQ=WK$. Complete the three rectangles as shown in Fig. II.

Sug. Consult 501 and 373.

504. Hence, the above demonstration shows that if the bases and altitudes of any two rectangles be measured by the same linear unit, the ratio of their respective products (see 500) will be the ratio of their respective surfaces; and consequently either may be assumed as the measure of the other.

For example, suppose the linear unit to be contained 5 times in AD, 18 times in DC, 3 times in HP, and 6 times in PN.

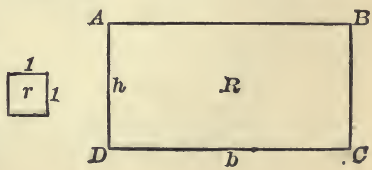
Then, $$A:a::5 \cdot 18:3 \cdot 6,$$

or $$\frac{A}{a}=\frac{90}{18}.$$

This simply means that, using a as the unit, the area A is 5 times as large as the area a; or that the area a, using A as the unit surface, is $\frac{1}{5}$ as large as A.

Practically it is more convenient to compare the areas of both rectangles with square of the linear unit (499) as a unit of surface. This comparison is formally shown in the enunciation and demonstration of the following theorem.

505. **The area of a rectangle is equal to the product of its base and altitude.**



Post. Let $ABCD$ be any rectangle; and in whatever linear unit the base and altitude be expressed, let r be a square whose sides are the

same unit. Designate its area by R, and its base and altitude by b and h respectively.

To Prove. $R = b \cdot h.$

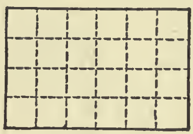
Dem. $R : r :: b \cdot h : 1 \cdot 1,$ (503)

or $$\frac{R}{r} = \frac{b \cdot h}{1 \cdot 1},$$

or, since r is the unit of surface

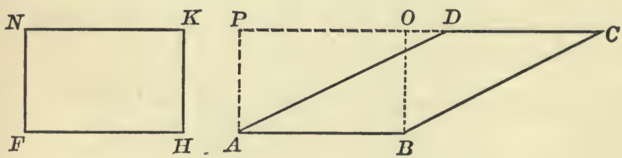
$$R = b \cdot h.$$ Q.E.D.

506. When the base and altitude are commensurable, this is rendered evident by dividing the rectangle into squares, each equal to the superficial unit, as shown in annexed figure. The above demonstration, however, includes the case when the base and altitude are incommensurable, hence,



507. If two rectangles have the same or equal bases and the same or equal altitudes, their areas are equal.

508. If a parallelogram and rectangle have the same or equal bases and the same or equal altitudes, they are equivalent.



Post. ——————.

To Prove. ——————.

Sug. Extend CD, draw BO and $AP \perp$ to AB, then compare $ABOP$ with $FHKN$ and $ABCD$.

509. The area of a parallelogram is equal to the product of its base and altitude.

510. If two parallelograms have the same or equal bases and the same or equal altitudes, they are equivalent.

511. The areas of any two parallelograms are in the same ratio as the products of their respective bases and altitudes.

512. If two parallelograms have the same or equal bases, their areas are in the same ratio as their altitudes.

513. If two parallelograms have the same or equal altitudes, their areas are in the same ratio as their bases.

514. If a triangle and parallelogram have the same or equal bases and the same or equal altitudes, the area of the latter is double that of the former.

Sug. Complete the parallelogram having the triangle for one half.

515. The area of a triangle is equal to one half the product of its base and altitude.

516. If two triangles have the same or equal bases and the same or equal altitudes, they are equivalent.

517. The areas of any two triangles are in the same ratio as the products of their respective bases and altitudes.

518. If two triangles have the same or equal bases, their areas are in the same ratio as their altitudes.

519. If two triangles have the same or equal altitudes, their areas are in the same ratio as their bases.

520. The area of a trapezoid is equal to the product of its altitude and one half the sum of its parallel sides.

521. The area of a rhombus is equal to one half the product of its diagonals.

522. The square described upon the sum of two lines is equivalent to the sum of the squares upon the two lines, plus twice the rectangle formed by the two lines.

523. The square described upon the difference of two lines is equivalent to the sum of the squares upon the two lines, minus twice the rectangle formed by the two lines.

524. The rectangle formed by the sum and difference of two lines is equivalent to the difference of the squares upon the two lines.

525. The square described upon the hypothenuse of a right triangle is equivalent to the sum of the squares described upon the legs.

Sch. This theorem was first demonstrated by Pythagoras, about 450 B.C., and hence is called the Pythagorean theorem. It has been a favorite one with mathematicians, and consequently about fifty different demonstrations of it have been recorded. Among the many diagrams used, the following are a few; and it is hoped that the pupil will endeavor to demonstrate it for himself, using either one of these, or, preferably, one of his own. Fig. I is a diagram used by Euclid in the demonstration of this theorem, constituting his famous "47th." Fig. VII is the diagram used by the late President Garfield, who was said to have utilized his leisure hours in Congress in mathematical investigations. In each figure ABC is the given triangle, and in Fig. VII ADC is one half the square upon the hypothenuse. DH is drawn parallel to BC to meet BA extended. The method is algebraic, and involves an equation between the sum of the areas of the three triangles and the trapezoid $BCDH$.

In Fig. IX consult 452 and 356. Then $AD = AB + CB$ and $AF = AB - CB$.

In Fig. XIII, PQ and HF are drawn parallel to AB, and ON parallel to AC. Points H and N, as also K and N, are joined. The lines NH and NK can then easily be proved to be one line. Then compare the areas AF and CF, also BN and AF. Compare also the areas PC and PB, AD and OC, then OC and PB.

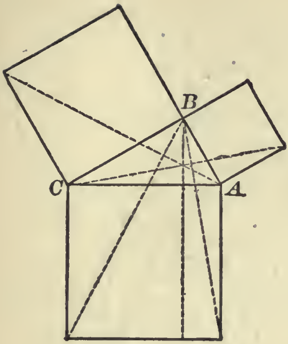


FIG. I.

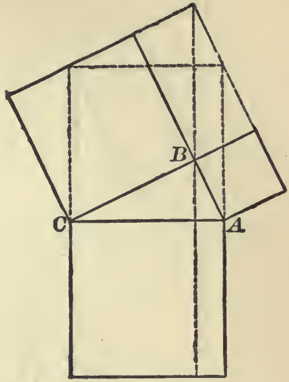


FIG. II.

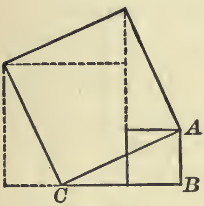


FIG. III.

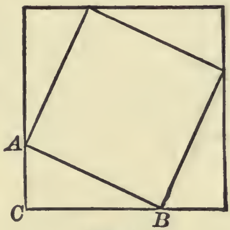


FIG. IV.

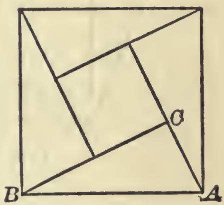


FIG. V.

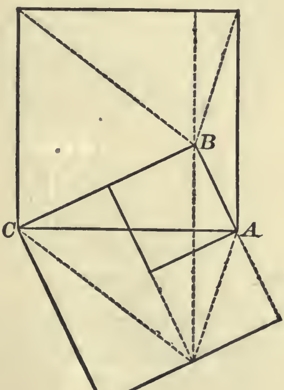


FIG. VI.

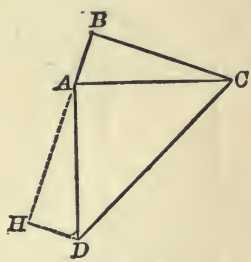


FIG. VII.

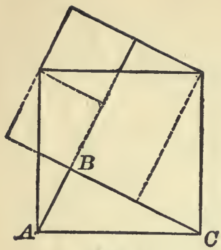


FIG. VIII.

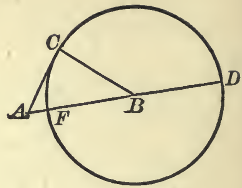


FIG. IX.

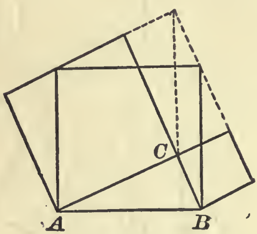


FIG. X.

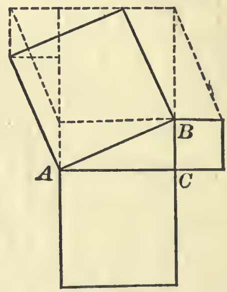


FIG. XI.

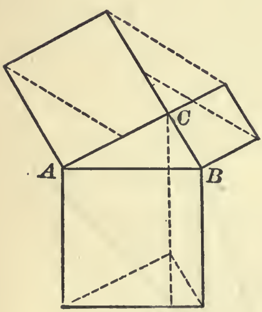


FIG. XII.

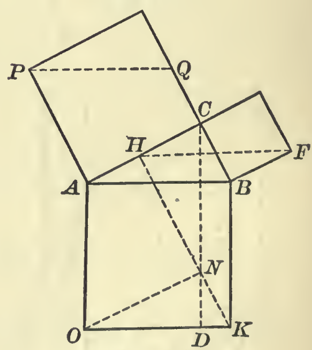
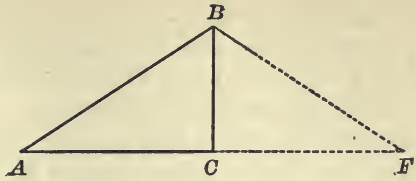


FIG. XIII.

526. If the square described on one side of a triangle is equivalent to the sum of the squares upon the other two sides, then the triangle is a right triangle.

Post. Let $\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$.



To Prove. $\angle ACB$ is a rt. $\angle$.

Cons. Draw $CF \perp$ to BC and equal to AC.

Sug. Consult 525, Ax. iii, and 95.

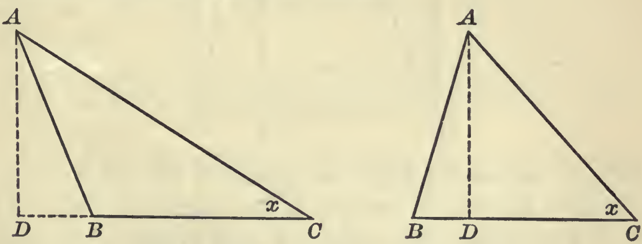
527. The square described upon the diagonal of a square is equivalent to twice the square of one of its sides.

528. The diagonal of a square is incommensurable with its side.

Sug. Prove it equal to $\sqrt{2}$.

529. In any triangle the square of the side opposite an acute angle is equal to the sum of the squares of the other two sides, minus twice the product of one of those sides and the projection of the other upon that side.

If C be the acute angle, then by a glance at the following diagram it will be readily seen that there may be two cases depending upon whether the projection involves an extension of one side, which will evidently be the case if the side to be projected is opposite an obtuse angle.



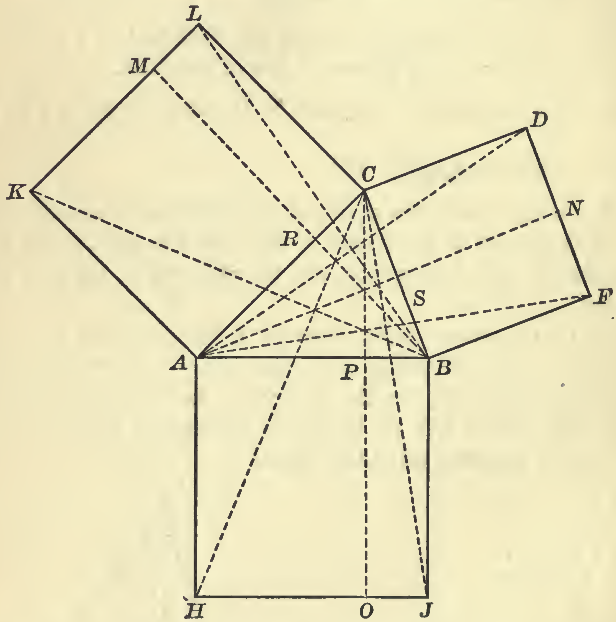
To Prove. $\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 - 2\,BC \cdot DC.$

In Case I. $DB = BC - DC.$

In Case II. $DB = DC - BC.$

The square of either of these two equations gives the same result; hence, to the square add the square of the perpendicular, and then combine the terms by using Theorem 525.

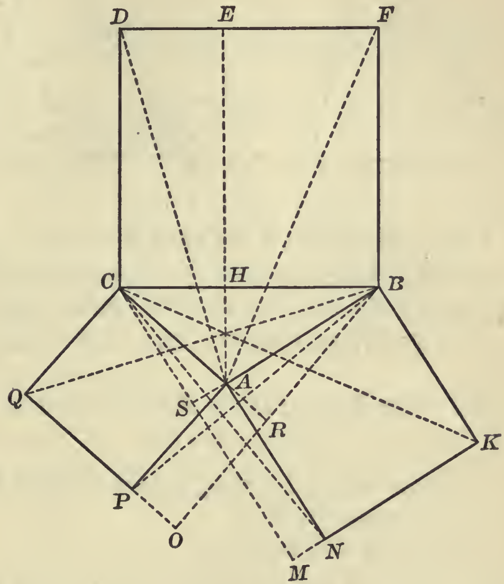
Sch. The above demonstration is algebraic and is given preference merely because of a less complex diagram. The diagram for the geometrical demonstration is herewith appended, however, with the requisite auxiliary lines drawn. The latter are drawn similar to those in 525.



530. In an obtuse triangle the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides, plus twice the product of one of those sides and the projection of the other upon that side.

Sug. Form an equation by placing the projection of the side opposite the obtuse angle equal to the sum of its two

parts, then proceed as in 529, or the following diagram may be used for a geometrical demonstration.



531. If any median of a triangle be drawn,

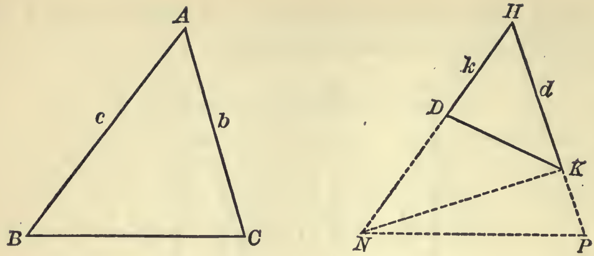
i. The sum of the squares of the other two sides is equal to twice the square of one half the bisected side, plus twice the square of the median; and

ii. The difference of the squares of the other two sides is equal to twice the product of the bisected side and the projection of the median upon that side.

Sug. Use 529 and 530, then combine the resulting equations.

532. If two triangles have an angle in each equal, their areas are in the same ratio as the products of the sides which include the equal angles.

Post. Let ABC and DHK be two triangles with $\angle A = \angle H$. Designate their areas by S and s, and the sides opposite the several angles by the corresponding small letters.



To Prove. $S : s :: c \cdot b :: k \cdot d$.

Cons. Extend the sides of one until they equal the corresponding sides of the other, as in the above diagram, and join NK.

Dem. Area DHK : Area NHK :: DH : NH. (519)

Area NHK : Area NHP :: HK : HP. Why?

Whence

Area DHK : Area NHP :: $DH \times HK$: $HN \times HP$. Why?

or $s : S :: k \cdot d : c \cdot b$. Q.E.D.

533. The areas of two similar triangles have the same ratio as the squares of any two homologous sides.

Sug. Consult 532, 372, and 362.

534. The areas of two similar triangles are in the same ratio as the squares of their homologous altitudes.

535. The areas of two similar polygons are in the same ratio as the squares of any two homologous sides.

Sug. Consult 477, 533, and 367.

536. The areas of two similar polygons are in the same ratio as the squares of any two homologous diagonals.

537. Any two homologous sides or diagonals of two similar polygons are in the same ratio as the square roots of their areas.

538. The areas of two regular polygons of the same number of sides are in the same ratio as the squares of any two homologous sides.

Sug. Consult 476.

539. The areas of two regular polygons of the same number of sides are in the same ratio as the squares of the radii of their circumscribed circles.

540. The areas of two regular polygons of the same number of sides are in the same ratio as the squares of their apothems.

541. The area of a regular polygon is equal to one half the product of its perimeter and apothem.

Sug. Draw radii and apothems, then consult 515.

ADVANCE THEOREMS.

542. In any triangle the product of any two sides is equal to the diameter of the circumscribed circle multiplied by the perpendicular drawn to the third side from the vertex of the angle opposite.

Sug. Construct the diameter from the same vertex as the perpendicular, and join its extremity with one of the other vertices, making a right triangle similar to the one formed by the perpendicular, whence the necessary proportion.

543. In any triangle the product of any two sides is equal to the product of the segments of the third side, formed by the bisector of the opposite angle, plus the square of the bisector.

Sug. Circumscribe a circle, and extend the bisector to the circumference, and connect its extremity with one of the other vertices of the triangle. If the vertices of the triangle be lettered A, B, and C, and the bisector be drawn from A, and D the point where the bisector crosses the side BC, and E the extremity of the bisector extended, then the triangles BAD and ACE can be proved similar.

544. If one leg of a right triangle is double the other, the perpendicular from the vertex of the right angle to the hypothenuse divides it into segments which are to each other as 1 to 4.

545. A line parallel to the bases of a trapezoid, passing through the intersection of the diagonals, and terminating in the non-parallel sides, is bisected by the diagonals.

546. In any triangle the product of any two sides is equal to the product of the segments of the third side, formed by the bisector of the exterior angle at the opposite vertex, minus the square of the bisector.

Sug. Circumscribe a circle as in 543.

547. The perpendicular from the intersection of the medians of a triangle, upon any straight line in the plane of the triangle, is one third the sum of the perpendiculars from the vertices of the triangle upon the same line.

548. If two circles are tangent to each other, their common tangent and their diameters form a proportion.

549. If two circles are tangent internally, all chords of the greater circle drawn from the point of contact are divided proportionally by the circumference of the smaller.

550. In any quadrilateral inscribed in a circle, the product of the diagonals is equal to the sum of the products of the opposite sides.

Sug. From one vertex draw a line to the opposite diagonal, making the angle formed by it and one side equal to the angle formed by the other diagonal and side which meets the former.

551. If three circles whose centers are not in the same straight line intersect one another, the common chords are concurrent.

552. If two chords be perpendicular to each other, the sum of the squares of the four segments is equal to the square of the diameter.

553. The sum of the squares of the diagonals of a quadrilateral is equal to twice the sum of the squares of the lines joining the middle points of the opposite sides.

554. If two triangles have two sides of one equal respectively to two sides of the other, and their included angles supplementary, the triangles are equivalent.

555. If a straight line be drawn through the center of a parallelogram, the two parts are equivalent.

556. If through the middle point of the median of a trapezoid a line be drawn, cutting the bases, the two parts are equivalent.

557. In every trapezoid the triangle which has for its base one leg, and for its vertex the middle point of the other leg, is equivalent to one half the trapezoid.

558. If any point within a parallelogram, selected at random, be joined to the four vertices, the sum of the areas of either pair of opposite triangles is equivalent to one half the parallelogram.

559. The area of a trapezoid is equal to the product of one of its legs and the distance of this leg from the middle point of the other.

560. The triangle whose vertices are the middle points of the sides of a given triangle is equivalent to one fourth the latter.

561. In any quadrilateral the sum of the squares of the four sides is equal to the sum of the squares of the diagonals, plus four times the square of the line joining the middle points of the diagonals.

Sug. Draw such lines as will utilize 531.

If the quadrilateral were a parallelogram, how would that modify the above theorem?

562. If two parallelograms have two contiguous sides respectively equal, and their included angles supplementary, the parallelograms are equivalent.

563. The lines joining the middle point of either diagonal of a quadrilateral to the opposite vertices, divide the quadrilateral into two equivalent parts.

564. The line which joins the middle points of the bases of a trapezoid divides the trapezoid into two equivalent parts.

PROBLEMS OF COMPUTATION.

565. The chord of one half a certain arc is 9 inches, and the distance from the middle point of this arc to the middle of its subtending chord is 3 inches. Compute the diameter of the circle.

566. The external segments of two secants to a circle from the same point are 3.5 m. and 25 dm. while the internal segment of the former is 150 cm. What is the internal segment of the latter?

567. The hypothenuse of a right triangle is 16 feet, and the perpendicular to it from the vertex of the opposite angle is 5 feet. Compute the values of the legs and the segments of the hypothenuse.

568. The sides of a certain triangle are 6, 7, and 8 hm. respectively. In a similar triangle the side corresponding to 8 is 40 hm. Compute the other two sides.

569. The sides of a certain triangle are 9, 12, and 15 feet respectively. Compute the segments of the sides made by the bisectors of the several angles.

570. If a vertical rod, 4 m. high, cast a shadow 150 cm. long, how high is a tree which, at the same time and place, casts a shadow 3.5 dkm. long.

571. The perimeters of two similar polygons are 200 and 300 feet respectively, and one side of the former is 24 feet. What is the corresponding side of the latter?

572. How long must a ladder be to reach a window 8 m. high, if the lower end of the ladder is 35 dm. from the side of the house?

573. Compute the lengths of the longest and the shortest chord that can be drawn through a point 6 inches from the center of a circle whose radius is 10 inches.

574. The distance from the center of a circle to a chord 10 dm. long is 12 dm. Compute the distance from the center to a chord 14 dm. long.

575. The radius of a circle is 5 inches. Through a point 3 inches from the center a diameter is drawn, and also a chord perpendicular to the diameter. Compute the length of this chord, and the distance from one end of the chord to the ends of the diameter.

576. Through a point 55 dm. from the center of a circle whose radius is 3.5 m. tangents are drawn. Compute the lengths of the tangents and of the chord joining the points of contact.

577. If a chord 8 feet long be 3 feet from the center of the circle, compute the radius and the distances from the end of the chord to the ends of the diameter which bisects the chord.

578. Through a point 500 mm. from the center of the circle whose radius is 13 dm. any chord is drawn. What is the product of the two segments of the chord? What is the length of the shortest chord that can be drawn through that point?

579. From the end of a tangent 20 inches long a secant is drawn through the center of a circle. If the exterior segment of this secant be 8 inches, what is the radius of the circle?

580. A tangent 7.2 dkm. long is drawn to a circle whose radius is 90 dm. Compute the exterior segment of a secant through the center from the extremity of the tangent.

581. The span of a roof is 28 feet, and each of its slopes measures 17 feet from the ridge to the eaves. Compute the height of the ridge above the eaves.

582. A ladder 13 m. long is placed so as to reach a window 8 m. high on one side of the street, and on turning the ladder over to the other side of the street it just reaches a window 10 m. high. What is the width of the street?

583. The bottom of a ladder is placed at a point 14 feet from a house, while its top rests against the house 48 feet from the ground. On turning the ladder over to the other side of the street, its top rests 40 feet from the ground. Compute the width of the street.

584. One leg of a right triangle is 3925 feet, and the difference between the hypothenuse and other leg is 625 feet. Compute the hypothenuse and the other leg.

585. Compute the area of a right isosceles triangle if the hypothenuse is 100 meters.

586. Compute the area of a rhombus if the sum of its diagonals is 12 dm. and their ratio is 3 : 5.

587. Compute the area of a right triangle whose hypothenuse is 13 feet and one of whose legs is 5 feet.

588. Compute the area of an equilateral triangle, one of whose sides is 40 hkm.

589. The area of a trapezoid is $3\frac{1}{8}$ acres; the sum of the two parallel sides is 242 yards. Compute the perpendicular distance between them.

590. The diagonals of a rhombus are 16.5 m. and 300 dm. respectively. Compute its area.

591. The diagonals of a rhombus are 88 feet and 234 feet respectively. Compute its area, and find length of one side.

592. The area of a rhombus is 354,144 sq. meters, and one diagonal is .672 km. Compute the other diagonal and one side.

593. The sides of a right triangle are in the ratio of 3, 4, and 5, and the altitude upon the hypothenuse is 20 yards. Compute the area.

594. Compute the area of a quadrilateral circumscribed about a circle whose radius is 25 m. and the perimeter of the quadrilateral is 400 m.

595. Compute the area of a regular hexagon inscribed in a circle whose radius is 12 cm.

596. The base of a triangle is 75 rods and its altitude 60 rods. Compute the perimeter of an equivalent rhombus if its altitude is 45 rods.

597. Upon the diagonal of a rectangle 4.5 m. by 25 dm. an equivalent triangle is constructed. Compute its altitude.

598. Compute the side of a square equivalent to a trapezoid whose bases are 56 feet and 44 feet respectively, and each of whose legs is 10 feet.

599. Find what part of the entire area of a parallelogram will be the area of the triangle formed by drawing a line from one vertex to the middle point of one of the opposite sides.

600. In two similar polygons two homologous sides are 15 feet and 25 feet respectively. The area of the first polygon is 450 square feet. Compute the area of the other polygon.

601. The base of a triangle is 32 km. and its altitude is 20 km. Compute the area of the triangle formed by drawing a line parallel to the base at a distance of 500 dkm. from the base.

602. The sides of two equilateral triangles are 20 and 30 yards respectively. Compute the side of an equilateral triangle equivalent to their sum.

603. If the side of one equilateral triangle is equivalent to the altitude of another, what is the ratio of their areas ?

604. The radius of a circle is 15 feet, and through a point 9 inches from the center any chord is drawn. What is the product of the two segments of this chord?

605. A square field contains 302,500 sq. m. Compute the length of fence that incloses it.

606. A square field 210 yards wide has a path around the inside of its perimeter which occupies just one seventh of the whole field. Compute the width of the path.

607. A street $3\frac{1}{2}$ km. long contains 5.6 hectars. How wide is the street?

608. The perimeter of a rectangle is 72 feet, and its length is twice its breadth. What is its area?

609. A chain 800 dm. long incloses a rectangle 15 m. wide. How much more area would it inclose if the figure were a square?

610. The perimeter of a square, and also of a rectangle whose length is four times its breadth, is 400 yards. Compute the difference in their areas.

611. A rectangle whose length is 25 m. is equivalent to a square whose side is 15 m. Which has the greater perimeter, and how much?

612. The perimeters of two rectangular lots are 102 yards and 108 yards respectively. The first lot is eight ninths as wide as it is long, and the second lot is twice as long as it is wide. Compute the difference in the value of the two lots at $ 1 per square foot.

613. A rhombus and a rectangle have equal bases and equal areas. Compute their perimeters if one side of the rhombus is 300 m. and the altitude of the rectangle is 22 dkm.

614. The altitudes of two triangles are equal, and their bases are 20 feet and 30 feet respectively. Compute the base of a triangle equivalent to their sum and having an altitude one fourth as great.

MEASUREMENT OF THE CIRCLE.—THEOREMS.

615. If radii of a regular circumscribed polygon of n sides be drawn, the chords joining successive points of intersection will form a regular polygon of n sides, and the sides of the two polygons will be parallel two and two.

616. If tangents to a circle be drawn parallel to the sides of a regular inscribed polygon of n sides, they will form a regular polygon of n sides.

617. If, in the same circle, a circumscribed and inscribed polygon of n sides have their sides parallel, two and two, their radii are congruent.

618. If two regular polygons of the same number of sides be inscribed in, or circumscribed about, the same circle, they are equal.

619. If tangents are drawn at the middle points of the arcs and terminating in the sides of a regular circumscribed polygon of n sides, a regular circumscribed polygon of $2n$ sides will thus be formed.

620. If the vertices of a regular inscribed polygon of n sides are joined to the middle points of the arcs subtended by the sides of the polygon, the lines thus drawn will form a regular inscribed polygon of $2n$ sides.

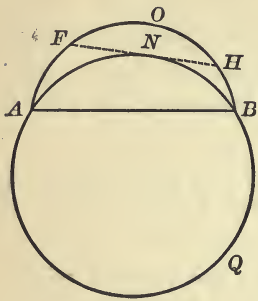
621. The perimeter of a regular inscribed polygon of n sides is less than the perimeter of the regular polygon of $2n$ sides inscribed in the same circle.

622. The perimeter of the regular circumscribed polygon of n sides is greater than the perimeter of the regular polygon of $2n$ sides circumscribed about the same circle.

623. If the extremities of a line be joined by the arc of a circle and by other lines which envelop this arc, the arc is the least of them all.

Post. Let AB be a chord of the circle ABQ, subtending the arc ANB, and let AOB be any one of the lines enveloping the arc ANB.

To Prove. Arc ANB is the least of them all.



Dem. AOB envelops ANB. (?)

$\therefore$ none of its points are within the circle, and all its points cannot be on ANB; *i.e.* some of its points must be outside ANB.

$\therefore$ from some point in ANB, as N, a tangent can be drawn terminating in AOB, as at F and H.

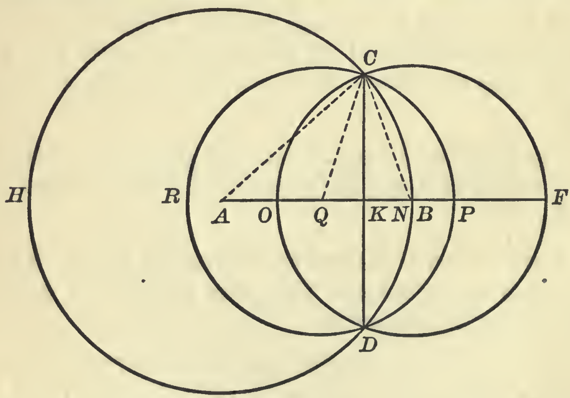
$$FH < FOH \quad (?)$$

$$\therefore AFNHB < AFOHB \quad (?)$$

$\therefore$ AOB is not the least line joining A and B. In like manner it can be shown that no enveloping line can be the least.

$\therefore$ ANB is the least. Q.E.D.

624. If two circles with unequal radii intersect each other, of the lesser arcs subtended by their common chord, the arc of the larger circle is less than the arc of the smaller circle.



Post. Let DCH and DCF be two intersecting circles of which DCH is the larger, and having the common chord CD.

To Prove. Arc $CND <$ arc COD.

Cons. Join their centers by line AB. Draw the radii AC and BC. Draw $CQ = CB$ and with Q as a center and radius QC construct circle DCR.

Dem. $AQ > AC - QC$ Why?

$\therefore$ circumference CDR will intersect QB between N and F as at P.

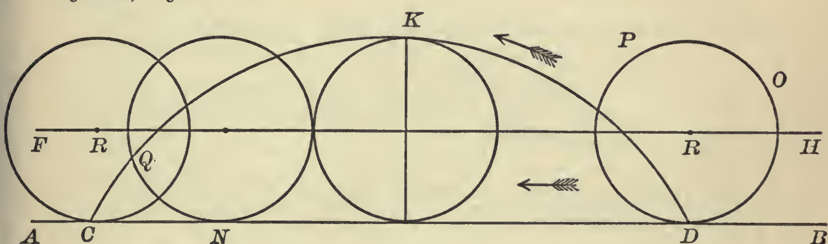
If $AQ = AC - QC$, they would be tangent to each other, and if $AQ < AC - QC$, one lies within the other, and both these results contradict the Hyp.

arc CPD = arc COD Why?

arc CND < arc CPD Why?

$\therefore$ arc CND < arc COD Why? Q.E.D.

625. If a circle be conceived to roll along a straight line without slipping and keeping always in the same plane, a point in the circumference will describe a curve called a *cycloid, e.g.*



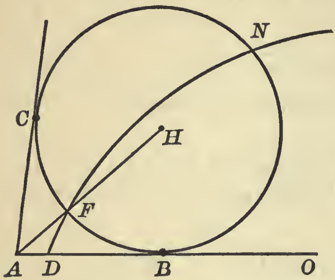
Suppose the circle DOP, which is tangent to line AB at D, to roll along the line AB without slipping, in the direction of the arrow and keeping always in the same plane. The center R will move in the line FH parallel to AB, while the point D will describe the curve DKC, which is called the *cycloid.*

It is evident from the definition

i. That the line CD is equal in length to the circumference of the circle.

ii. That the line CN is equal in length to the arc QN.

626. If, from a point without a circle, two tangents are drawn, the lesser of the two arcs joining the points of contact is less than the sum of the two tangents.



Post. Let CBN be a circle, H its center, and AC and AB two tangents to that circle drawn from point A, the points of contact being C and B.

To Prove. That arc CB is less than $AC + AB$.

Cons. Suppose AH to be drawn, and let the curve DFN be the cycloid traced by the point F as the circle rolls along the line OA toward A.

Dem.	$DB < AB$	Why?	$\therefore$ Arc	$CF < AC$	Why?
Arc	$FB = DB$	See 625.	$\therefore$ Arc	$FB +$ arc $CF < AC + AB$	Why?
$\therefore$ Arc	$FB < AB$	Why?	or Arc	$CB < AC + AB$	
Arc	$FB =$ arc CF	Why?			Q.E.D.
	$AC = AB$	Why?			

627. The circumference of a circle is greater than the perimeter of any polygon inscribed in it.

628. The circumference of a circle is less than the perimeter of any polygon circumscribed about it.

629. The area of a regular inscribed polygon of 2 n sides is greater than the area of a regular polygon of n sides inscribed in the same circle.

630. The area of a regular circumscribed polygon of 2 n sides is less than the area of a regular polygon of n sides circumscribed about the same circle.

631. If the number of sides of a regular inscribed polygon be continuously doubled, its perimeter will become an increasing variable and approach the circumference as its limit.

632. If the number of sides of a regular inscribed polygon be continuously doubled, its apothem will become an increasing variable, and approach the radius as its limit.

633. If the number of sides of a regular circumscribed polygon be continuously doubled, its perimeter will become a decreasing variable, and approach the circumference as its limit.

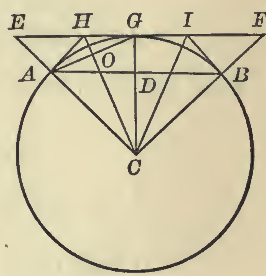
634. If the number of sides of a regular circumscribed polygon be continuously doubled, its radius will become a decreasing variable, and approach the radius of the circle as its limit.

635. If the number of sides of a regular inscribed polygon be continuously doubled, its area will become an increasing variable and approach the area of the circle as its limit.

636. If the number of sides of a regular circumscribed polygon be continuously doubled, its area will become a decreasing variable and approach the area of the circle as its limit.

637. The area of a regular inscribed polygon of 2 n sides is a mean proportional between the areas of two polygons, each of n sides, one inscribed within, and the other circumscribed about, the same circle.

Post. Let AB be one side of a regular inscribed polygon of n sides, and EF one side of a regular circumscribed polygon of n sides and parallel to AB. Let CE and CF be radii of this polygon. Also let CG be a radius of the circle drawn to point of contact. Let AH and BI be tangents and AG a chord of the circle. Let also CH and CI be drawn. Designate the area of the polygon whose side is AB by p, that of the polygon whose side is EF by P, and that whose side is AG by p'.



To Prove. $p : p' :: p' : P.$

Dem. $2n \cdot \overset{\text{Area.}}{ACD} = p$, $2n \cdot \overset{\text{Area.}}{ACG} = p'$, and $2n \cdot \overset{\text{Area.}}{ECG} = P$.

$$\text{Area } ACD : \text{area } ACG :: CD : CG \quad (?)$$
$$\text{Area } ACD : \text{area } ACG :: p : p' \quad (?)$$
$$\therefore\ p : p' :: CD : CG \quad (?)$$
$$\text{Area } CAG : \text{area } CEG :: CA : CE \quad (?)$$
$$\text{Area } CAG : \text{area } CEG : p' : P \quad (?)$$
$$\therefore\ CA : CE :: p' : P \quad (?)$$
$$CA : CE :: CD : CG \quad (?)$$
$$\therefore\ p : p' :: p' : P \quad (?) \text{ Q.E.D.}$$

638. **If polygons of n and $2n$ sides be inscribed in, and circumscribed about, the same circle, the area of the circumscribed polygon of $2n$ sides equals twice the product of the areas of the polygons of n sides divided by the sum of the areas of the two inscribed polygons.**

Post. Use same diagram as in previous theorem, and designate areas the same, also designate area of the polygon whose side is HI by P'.

To Prove. $$P' = \frac{2pP}{p+p'}.$$

Dem. $$\text{Area } CGH : \text{area } CHE :: GH : HE \quad (?)$$
$$CG : CE :: GH : HE \quad (?)$$
$$\therefore\ \text{Area } CGH : \text{area } CHE :: CG : CE \quad (?)$$

From the Dem. in 637, 3d proportion

$$p : p' :: CD : CG$$
$$CG : CE :: CD : CA \quad (?)$$

or $$CG : CE :: CD : CG \quad (?)$$
$$\therefore\ p : p' :: CG : CE \quad (?)$$
$$\therefore\ \text{Area } CGH : \text{area } CHE :: p : p' \quad (?)$$
$$\therefore\ \text{Area } CGH : \text{area } CGE :: p : p + p' \quad (?)$$
$$\therefore\ 2 \text{ Area } CGH : \text{area } CGE :: 2p : p + p' \quad (?)$$

or $$\text{Area } CHI : \text{area } CGE :: 2p : p + p' \quad (?)$$

$$2\,n \cdot \overset{\text{Area.}}{CHI} = P', \text{ and } 2\,n \cdot \overset{\text{Area.}}{CGE} = P \qquad (?)$$

$$\therefore \text{ Area } CHI : \text{area } CGE :: P' : P \qquad (?)$$

$$\therefore P' : P :: 2\,p : p + p' \qquad (?)$$

$$\therefore P' = \frac{2\,pP}{p + p'}. \qquad (?) \text{ Q.E.D.}$$

639. The chords which join the extremities of two perpendicular diameters form a square.

640. The diagonals of an inscribed square are diameters of the circumscribed circle.

641. The tangents to a circle whose points of contact are the vertices of an inscribed square will form a square.

642. The area of an inscribed square is equal to twice the square of the radius.

643. The area of a circumscribed square is equal to four times the square of the radius.

644. *Problem.* To compute the areas of the inscribed and circumscribed polygons of 2 n sides, having given the areas of those of n sides. From 637 and 638 we have

$$\text{I. } p' = \sqrt{p \cdot P} \text{ and II. } P' = \frac{2\,pP}{p + p'}.$$

Letting p and P represent the area of inscribed and circumscribed squares respectively and designating the radius by r, we have from 642 and 643

$$p = 2\,r^2 \text{ and } P = 4\,r^2.$$

By substituting these values in I., we have

$$p' = 2.82843\,r^2.$$

Then by substituting values of p, p', and P in II., we have

$$P' = 3.31371\,r^2.$$

Thus we have computed the areas of the regular inscribed and circumscribed octagons in terms of the radius of the circle.

Again, calling these p and P, p' and P' will be polygons of sixteen sides; and using the formulæ as before, the areas of the latter may be computed. Repeating the process, those of thirty-two, sixty-four, etc., sides may be, in like manner, computed.

Below are the tabulated results to seven decimal places for thirteen doublings of the number of sides of the polygons:

Number of Sides.	Area of Inscribed Polygon.	Area of Circumscribed Polygon.
4	$2.0000000\, r^2$	$4.0000000\, r^2$
8	$2.8284271\, r^2$	$3.3837085\, r^2$
16	$3.0614675\, r^2$	$3.1825979\, r^2$
32	$3.1214452\, r^2$	$3.1517240\, r^2$
64	$3.1365485\, r^2$	$3.1441184\, r^2$
128	$3.1403312\, r^2$	$3.1422236\, r^2$
256	$3.1412773\, r^2$	$3.1417504\, r^2$
512	$3.1415138\, r^2$	$3.1416321\, r^2$
1024	$3.1415729\, r^2$	$3.1416025\, r^2$
2048	$3.1415877\, r^2$	$3.1415951\, r^2$
4096	$3.1415914\, r^2$	$3.1415933\, r^2$
8192	$3.1415912\, r^2$	$3.1415928\, r^2$
16384	$3.1415925\, r^2$	$3.1415927\, r^2$
32768	$3.1415926\, r^2$	$3.1415926\, r^2$

Hence, since the area of the circle is greater than that of the inscribed polygon and less than that of the circumscribed polygon (635 and 636), $3.1415926\, r^2$ must be the area of the circle correct to within less than one ten-millionth part of r^2. But by continuing the process, the areas of the two polygons may be made to agree to any desired number of decimal places, and, therefore, such result may be taken as the area of the circle without sensible error. If r be taken as unity, it would, of course, vanish from the expression, and consequently 3.1415926 may be taken as the area of the circle whose radius is unity, correct to seven decimal places.

645. If in 637 and 638 we let P, P', p, and p' represent perimeters instead of areas, then,

$$\text{I. } P'=\frac{2\,pP}{p+P} \text{ and II. } p'=\sqrt{p\times P'}.$$

Dem.

$P:p::EC:AC$ (or CG) (?)

$EH:HG::EC:CG$ (?)

$\therefore P:p::EH:HG$ (?)

$\therefore P+p:2p::EH+HG:2HG$ (?)

$\therefore P+p:2p::EG:HI$ (?)

But

$P:P'::EG:HI$ (?)

$\therefore P+p:2p::P:P'$ (?)

$\therefore P'=\dfrac{2\,pP}{p+P}$ I. (?)

Again

$AD:AG::p:p'$ (?)

and

$AG:HI::p':P'$ (?)

$\therefore \dfrac{AG}{2}:\dfrac{HI}{2}::p':P'$ (?)

$OG:HG::AD:AG$ (?)

$OG=\dfrac{AG}{2}$, and $HG=\dfrac{HI}{2}$ (?)

$\therefore \dfrac{AG}{2}:\dfrac{HI}{2}::AD:AG$ (?)

$\therefore AD:AG::p':P'$ (?)

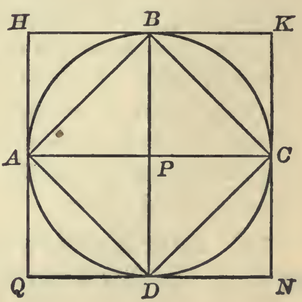
$\therefore p:p'::p':P'$ (?)

$\therefore p'=\sqrt{p\cdot P'}$ (?)

Q.E.D.

646. We will now use these formulæ for the computation of perimeters in a manner similar to that in 644, beginning with the circumscribed and inscribed squares.

The perimeter of the circumscribed square in terms of the diameter, calling the latter D, is, obviously, $4\,D$. In the right triangle APD



$$\overline{AD}^2=\overline{AP}^2+\overline{PD}^2=2\,\overline{AP}^2=2\,r^2$$
$$=2\left(\frac{D}{2}\right)^2=\frac{D^2}{2}$$

$$\therefore AD=\frac{D}{\sqrt{2}},\quad 4\,AD=\frac{4}{\sqrt{2}}D=D\sqrt{8}.$$

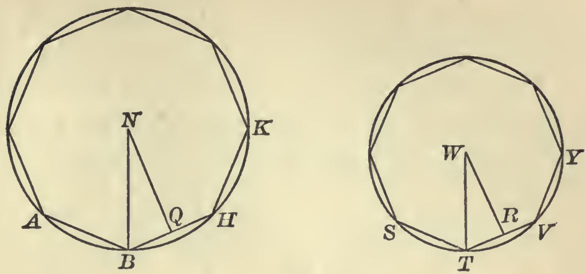
Hence the perimeter of the in-

scribed square is 2.8284271 D. Continuing as in 644 and tabulating the results, we have the following:

Number of Sides.	Perimeter of Inscribed Polygon.	Perimeter of Circumscribed Polygon.
4	2.8284271 D	4.0000000 D
8	3.0614675 D	3.3137085 D
16	3.1214452 D	3.1825979 D
32	3.1365485 D	3.1517249 D
64	3.1403312 D	3.1441184 D
128	3.1412773 D	3.1422236 D
256	3.1415138 D	3.1417504 D
512	3.1415729 D	3.1416321 D
1024	3.1415877 D	3.1416025 D
2048	3.1415914 D	3.1415951 D
4096	3.1415923 D	3.1415933 D
8192	3.1415924 D	3.1415928 D
16384	3.1415925 D	3.1415927 D
32768	3.1415926 D	3.1415926 D

Hence, since the circumference of the circle is greater than the perimeter of the inscribed polygon, and less than that of the circumscribed polygon (627 and 628), 3.1415926 D must be the circumference of the circle correct to within less than one ten-millionth part of D. But by continuing the process the perimeters of the two polygons may be made to agree to any desired number of decimal places, and therefore such result may be taken as the circumference of the circle without sensible error. If D be taken as unity, it would, of course, vanish from the expression, and consequently 3.1415926 may be taken as the circumference of a circle whose diameter is unity, correct to seven decimal places.

647. The circumferences of two circles are in the same ratio as their radii, and also their diameters.



Post. Let $ABHK$, etc., and $STVY$, etc., be two circles whose centers are N and W, and designate the circumferences by C and c, and their radii by R and r, and their diameters by D and d, the capital letter referring to the larger circle.

To Prove. I. $C : c :: R : r$

II. $C : c :: D : d$

Cons. Inscribe in each a regular polygon of n sides, and construct the radii NB and WT, and the apothems NQ and WR.

Dem. Designating the perimeters by P and p,

$$P : p :: NB : WT \qquad (?)$$

$$\therefore\ P : NB :: p : WT \qquad (?)$$

or

$$\frac{P}{NB} = \frac{p}{WT}. \qquad (?)$$

We will now inscribe polygons with double the number of sides, and continue this process indefinitely; then P and p become increasing variables approaching C and c as their limits.

$\because\ \frac{P}{NB}$ becomes an incr. var. appr. its limit $\frac{C}{NB}$ and $\frac{p}{WT}$ becomes an incr. var. appr. its limit $\frac{c}{WT}$.

$$\therefore\ \frac{C}{NB} = \frac{c}{WT}, \text{ or } \frac{C}{R} = \frac{c}{r}; \quad (?)$$

i.e.

$$C : R :: c : r$$

$$\therefore\ C : c :: R : r \qquad (?)$$

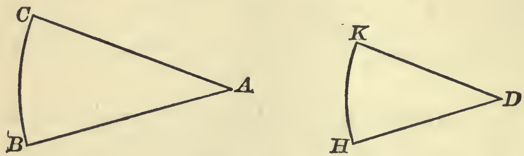
$$\therefore\ C : c :: D : d \qquad (?)$$

Q.E.D.

648. The areas of two circles are in the same ratio as the squares of their radii and of their diameters.

Sug. Use method similar to the above, and consult 540.

649. Similar arcs are in the same ratio as the radii of the circumferences of which they are a part, and also as the diameters.



Post. Let CB and KH be two similar arcs, and A and D the centers of the circumferences of which they are a part.

Cons. Draw the radii AC, AB, DK, and DH.

(Designate circumferences, diameters, and radii, as before.)

To Prove. I. Arc CB : arc KH : : $R : r$

II. Arc CB : arc KH : : $D : d$

Dem. $\angle A = \angle D$. Why?

Hence arc CB is the same part of the circumference C as arc KH is of the circumference c.

$\therefore$ Arc CB : arc KH : : $C : c$ (?)

But $R : r : : C : c$ (?)

$\therefore$ I. Arc CB : arc KH : : $R : r$ (?)

II. Arc CB : arc KH : : $D : d$ Q.E.D.

650. The areas of similar sectors are in the same ratio as the squares of the radii, and also of the diameters, of the circles of which they are a part.

651. The area of a circle is equal to one half the product of its circumference and radius.

Sug. Consult 541.

652. The area of a sector is equal to one half the product of its arc and radius.

653. The areas of similar segments are in the same ratio as the squares of their radii, the squares of their diameters, and as the squares of their chords.

Sug. Consult Theorems 650 and 533 and utilize the fact that the area of the segment is equal to the area of the sector minus the area of the triangle. For brevity, designate the areas of the triangles by T and t, the areas of the sectors by K and k, and the areas of the segments by S and s.

654. I. Let us designate the circumference of a circle whose diameter is unity by π, and the circumference of any other circle by C; its diameter by D; its radius by R; and its area by A.

Then $$C : \pi :: D : 1. \qquad (?)$$

Hence $$\text{I. } C = \pi D,$$

whence $$\text{II. } \frac{C}{D} = \pi,$$

or $$\text{III. } C = \pi 2 R.$$

Multiplying both members of this equation by $\frac{R}{2}$, we have,

$$\frac{CR}{2} = \pi R^2.$$

But, by 651, $\frac{CR}{2} =$ the area of the circle; hence

$$\text{IV. } A = \pi R^2.$$

Hence the area of any circle is equal to the square of its radius multiplied by the constant quantity π, and the circumference of every circle is equal to the product of its diameter (or twice its radius) by the same quantity π.

From II. above, it is readily seen that π is the ratio of the circumference of any circle to its diameter, or of a semicircumference to its radius.

The exact numerical value of π can be only approximately expressed. As computed in 646 it is 3.1415926, but for

practical purposes in computing its value is usually taken as 3.1416. In many cases 3.14, or $\frac{22}{7}$, is sufficiently accurate.

The symbol π is the first letter of the Greek word meaning perimeter, or circumference.

II. The **quadrature of the circle** is the problem which requires the finding of a square which shall be equal in area to that of a circle with a given radius. Now since the area of a circle is equal to its circumference multiplied by one half its radius, if a straight line of same length as circumference be taken as the base of a rectangle, and one half the radius as its altitude, their product will be the area of the rectangle, also of the circle. It is also evident that, if a line which is a mean proportional between these two be taken as the side of a square, the area of this square will be equal to that of the rectangle, and consequently to that of the circle. It will be shown in the series of problems of construction (Prob. 837) how this mean proportional can be found; hence, to "square the circle" we must be able to find the circumference when the radius is known, or vice versa. For accomplishing this, we must know the ratio of the circumference to its diameter or radius. But this ratio, as has been remarked before, can be only approximately expressed; for, as the higher mathematics prove, the circumference and diameter are incommensurable, but the approximation has been carried so far that the error is infinitesimal. Archimedes, about 250 B.C., was the first to assign an approximate value to π. He found that it must be between 3.1428 and 3.1408. In 1540 Metius computed it correctly to 6 places. Later, in 1579, Vieta carried the approximation to 10 places, Van Ceulen to 36 places, Sharp to 72 places, Machin to 100 places, De Lagny to 128 places, Rutherford to 208 places, and Dr. Clausen to 250 places. In 1853 Rutherford carried it to 440 places, and in 1873 Shanks computed it to 707 places, but these latter results do not appear to have been verified. The following is its value to 208 places, as computed by Rutherford. It should be stated, however, that these

extensive computations are by the more expeditious methods of trigonometry.

$$
\begin{aligned}
\pi = 3.&14159265358979323846264338327\text{-}\\
&50288419716939937510582097494\text{4-}\\
&59230781640628620899862803482\text{5-}\\
&34271170679821480865132823066\text{47-}\\
&09384460955058223172535940812\text{8-}\\
&48473781392038633830215747399\text{6-}\\
&\quad 008259312591294018328065174\text{4+}
\end{aligned}
$$

Some idea of the accuracy of the above value may be formed from the following statement taken from Peacock's Calculus: "If the diameter of the universe be 100,000,000,000 times the distance of the sun from the earth (about 93,000,000 miles), and if a distance which is 100,000,000,000 times this diameter be divided into parts, each of which is one 100,000,000,000th part of an inch; then if a circle be described whose diameter is 100,000,000,000 times as often as each of those parts of an inch is contained in it; then the error in the circumference of this circle, as computed from this approximation, will be less than 100,000,000,000th part of the one 100,000,000,000th part of an inch."

ADVANCE THEOREMS.

655. In two circles of different radii, angles at the center subtending arcs of equal lengths are to each other inversely as the radii.

656. If, from any point within a regular polygon of n sides, perpendiculars be drawn to all the sides, the sum of these perpendiculars is equal to n times the apothem.

657. An equiangular polygon inscribed in a circle is regular if the number of its sides be odd.

658. An equilateral polygon circumscribed about a circle is regular if the number of its sides is odd.

659. The sum of the squares of the lines joining any point in the circumference of a circle with the vertices of an inscribed square, is equal to twice the square of the diameter of the circle.

660. The area of the ring included between two concentric circles is equal to the area of the circle whose diameter is that chord of the outer circle which is tangent to the inner.

661. If three circles be described upon the sides of a right triangle as diameters, the area of that described upon the hypothenuse is equal to the sum of the areas of the other two.

662. If upon the legs of a right triangle semicircumferences are described outwardly, the sum of the areas contained between these semicircumferences and the semicircumference passing through the three vertices is equal to the area of the triangle.

663. If a semicircle be described on the chord of a quadrant as a diameter, the area of the crescent is equal to that of the triangle of the quadrant.

664. If the diameter of a circle be divided into two parts, and upon these parts semicircumferences are described on opposite sides of the diameter, these semicircumferences will divide the circle into two parts, which have the same ratio as the two parts of the diameter.

665. If two chords of a circle be perpendicular to each other, the sum of the areas of the four circles described upon the four segments as diameters will be equal to the area of the given circle.

666. If squares be constructed outwardly upon the sides of a regular hexagon, their exterior vertices will be the vertices of a regular dodecagon.

667. The radius of a regular inscribed polygon is a mean proportional between its apothem and the radius of a regular polygon of the same number of sides circumscribed about the same circle.

668. The area of a regular dodecagon is equal to three times the square of its radius.

669. If the radius of a circle be divided in extreme and mean ratio, the larger part will be equal to the side of a regular decagon inscribed in the same circle.

670. The apothem of a regular inscribed pentagon is equal to one half the sum of the radius of the circle and the side of the regular decagon inscribed in the same circle.

671. The square of the side of a regular inscribed pentagon is equivalent to the sum of the squares of the side of the regular inscribed decagon and the radius of the circle.

PROBLEMS OF COMPUTATION.

672. Compute the side of a regular circumscribed trigon in terms of the side of the regular trigon inscribed in the same circle. Compare their areas.

673. Compute the side of a circumscribed square in terms of the side of the square inscribed in the same circle.

674. Compute the apothem of a regular inscribed trigon in terms of the side of the regular hexagon inscribed in the same circle.

675. Compute the apothem of a regular inscribed hexagon in terms of the side of the regular trigon inscribed in the same circle.

676. Regular trigons and hexagons are both inscribed in, and circumscribed about, the same circle. Compare their areas.

677. Compute the area of a regular polygon of 12 sides inscribed in a circle whose radius is 50 cm.

678. Compute the perimeter of a regular pentagon inscribed in a circle whose radius is 10 feet.

679. The perimeter of a regular hexagon is 480 m., and that of a regular octagon is the same. Which has the greater area, and how much?

680. If paving blocks are in the shape of regular polygons (*i.e.* their cross-sections), how many shapes can be employed in order to completely fill the space?

681. Compute the diameter of a circle whose circumference is 225 dm.

682. The diameter of a carriage wheel is 4 feet and 3 inches. How many revolutions does it make in traversing one fourth of a mile?

683. What is the width of a ring between two concentric circumferences whose lengths are 480 cm. and 3.6 m.?

684. Find the length of an arc of 36° in a circle whose diameter is 36 inches.

685. In raising water from the bottom of a well by means of a wheel and axle it was found that the axle, whose diameter was 20 cm., made 20 revolutions in raising the bucket. Compute the depth of the well.

686. Find the central angle subtending an arc 6 feet and 4 inches long, if the radius of the circle be 8 feet and 2 inches.

687. If the radius of a circle is 6 meters, find the perimeter of a sector whose angle is 45°.

688. If the central angle subtending an arc 10 feet and 6 inches long is 72°, what is the length of the radius of the circle?

689. If the length of the meridian of the earth be 40,000,000 meters, what is the length of an arc of $1''$?

690. Two arcs have the same angular measure, but the length of one is twice that of the other. Compare the radii of those arcs.

691. Compute the area of a circle whose circumference is 100 dkm.

692. Two arcs have the same length, but their angular measurements are 20° and 30° respectively. If the radius of the first arc is 6 feet, compute the radius of the other.

693. Find the circumference of a circle in meters whose area is 2.5 hektares.

694. The diameter of a circle is 40 feet. Find the side of a square which is double the area of the circle.

695. The area of a square is 196 square meters. Find the area of a circle inscribed in the square.

696. A circular fish-pond which covers an area of 5 acres and 100 square rods is surrounded by a walk 5 yards wide. Compute the cost of graveling the walk at $6\frac{1}{4}$ cents per square yard.

697. What must be the width of a walk around a circular garden containing $1\frac{3}{4}$ hektares, in order that the walk may contain exactly one fourth of a hektare?

698. A carpenter has a rectangular piece of board 15 inches wide and 20 inches long, from which he wishes to cut the largest possible circle. How many square inches of the board must he cut away?

699. The perimeter of a circle, a square, and a regular trigon are each equal to 144 kilometers. Compare their areas.

700. If the radius of a circle be 12 inches, what is the radius of a circle 10 times as large?

701. What will it cost to pave a circular court 240 dkm. in diameter at $4.75 per square meter, leaving in the center a hexagonal space, each side of which measures 2.5 meters.

702. A circle 18 feet in diameter is divided into three equivalent parts by two concentric circumferences. Find the radii of these circumferences.

703. If the chord of an arc be 720 dm., and the chord of its half be 38 meters, compute the diameter of the circle.

704. The chord of half an arc is 17 feet, and the height of the arc is 7 feet. Compute the diameter of the circle.

705. The lengths of two chords, drawn from the same point in the circumference of a circle to the extremities of a diameter, are 6 feet and 8 feet respectively. Compute the area of the circle.

706. The area of a sector is 16.5 ares, and the angle of the sector is 36°. Compute the radius of the circle and perimeter of the sector.

707. Compute the area of a circle in which the chord, 3 feet long, subtends an arc of 120°.

708. Compute the area of a segment whose arc is 300°, the radius of the circle being 50 cm.

709. The areas of two concentric circles are as 5 to 8. The area of that part of the ring which is contained between two radii making the angle 45° is 300 square feet. Compute the radii of the two circles.

710. What is the altitude of a rectangle equivalent to a sector whose radius is 15 m., if the base of the rectangle is equal to the arc of the sector?

711. Compute the radius of a circle, if its area is doubled by increasing its radius one foot.

712. The radius of a circle is 400 cm. Through a point exterior to the circle two tangents are drawn, making an angle of 60°. Compute the area of the figure bounded by the tangents and the intercepted arc.

713. Three equal circles are drawn tangent to each other, with a radius of 12 feet. Compute the area contained between the circles.

714. Upon each side of a square, as a diameter, a semi-circumference is described within the square. If the side of the square is 10 meters, compute the sum of the areas of the four leaves in ares.

715. In a circle whose radius is 100 feet two parallel chords are drawn on the same side of the center, one equal to the side

of a regular hexagon, and the other to the side of a regular trigon, both inscribed in the given circle. Compute the area of the circle comprised between the two parallel chords.

716. A quarter-mile running track consists of two parallel straight portions joined together at the ends by semicircumferences. The extreme length of the plot inclosed by the track is 180 yards. Compute the cost of sodding this plot at 25 cents per square yard.

717. The perimeter of a certain church window is made up of three equal semicircumferences, the centers of which form the vertices of an equilateral triangle whose sides are 1 m. long. Compute the area and perimeter of the window.

718. Two flower beds have equal perimeters. One of the beds is circular, and the other has the form of a regular hexagon. The circular bed is closely surrounded by a walk 7 feet wide. The area of the walk is to that of the bed as 7 to 9. Compute the diameter of the circular bed and the area of the hexagonal bed.

719. A crescent-shaped region is bounded by a semicircumference of radius a, and another circular arc, the center of the circumference of which lies on the semicircumference produced. Compute the area and perimeter of the crescent region.

720. Six coins, each of radius a, are placed close together on a table so that their centers are at the vertices of a regular hexagon. Compute the area and perimeter of the inclosed figure.

721. Every cross-section of the train house of a railway station has the form of a pointed arch made of two circular arcs, the centers of which are on the ground. The radius of each arc is equal to the width of the building, 210 feet. Compute the distance across the building measured over the roof, and show that the area of the cross-section is $3675(4\pi - 3\sqrt{3})$ square feet.

PROBLEMS OF CONSTRUCTION.

722. In the demonstration of the foregoing theorems it has been assumed that certain constructions were possible; *i.e.* that perpendiculars and parallels could be drawn, that lines and angles could be bisected, etc. It is now proposed to show that those and many other problems can be performed, so our previous demonstrations are not vitiated that are in any way dependent upon such constructions.

The solution of a geometrical problem of construction involves in general three steps, viz.:

i. The *construction* proper, by the use of the compass and ruler.

ii. *Demonstration* to prove the correctness of the construction.

iii. *Discussion* of its limitations and applications, including the number of possible constructions.

If numerical or algebraical results are also required, then there is, of neccssity,

iv. *Computation*, by which numerical values are ascertained, involving also the use of general symbols in obtaining algebraic formulæ.

Each pupil should be provided with a good pair of compasses, for the use of either pen and ink, or pencil, a good ruler with straight edges, besides one hard and one soft pencil. The graphite of the pencil should be sharpened flat, so that a fine line can be made.

PROBLEMS.

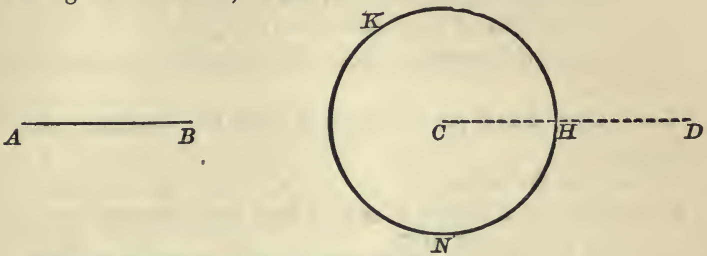
723. It is required to find a point which is a given distance from a given point.

Post. Let C be the given point, and AB the given distance.

We are required to find a point which shall be at the distance AB from C.

Cons. First, with the ruler, from C draw an indefinite straight line, as CD, in any direction. Then with the com-

passes, using C as a center and AB as a radius, draw an arc cutting the line CD, as at H.



Then H is the required point; for,

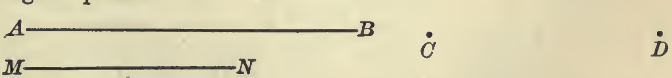
Dem. If, in applying the compasses to AB a circle had been constructed, and the circle HKN also completed, then the circles would have equal radii.

$\therefore$ H is the same distance from C that A is from B.

Discussion. Since, from the definition of a circle, all points in the circumference are equidistant from the center, it follows that every point in the circumference HKN is the same distance from C that A is from B. Consequently any point in that circumference answers the conditions of the problem. Q.E.F.

724. *Def.* Whenever a line is found such that any point in it selected at random will fulfill certain specified conditions, or such that all points in it have a common property, that line is called the *locus* of that point or points. Hence we may say in the above case, that the circumference HKN is the locus of the point which is the distance AB from point C, or, as some prefer to put it, it is the locus of all points which are at the distance AB from point C.

725. It is required to find the point, having given its distances from two given points.



Post. Let the two given distances be AB and MN, and the two given points C and D.

We are required to find a point which is AB distant from C, and MN distant from D, or vice versa.

Sug. Find locus of the point which is AB distant from point C, and locus of the point which is MN distant from point D, and vice versa.

Dis. How many points, then, satisfy the condition of the problem?

Determine the result if

i. the distance between C and D had been greater than the sum of AB and MN.

ii. If it had been equal to the sum of AB and MN.

iii. If it had been equal to the difference of AB and MN.

iv. If it had been less than the difference of AB and MN.

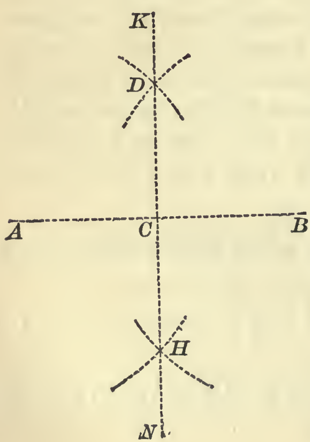
(See Theorem 322.)

726. It is required to find the point which is equidistant from two given points.

Post. Let A and B be the two given points.

We are to find a point which is the same distance from A as from B.

It is evident that whether there be more or not, there must at least be one midway between A and B; so join AB.



With A and B as centers, construct, with the same radius, two circles which shall intersect. How can you tell whether or not they will intersect?

Connect the points of intersection, as D and H. Then DH sustains what relation to the two circles?

The line AB sustains what relation to the two circles?

Then what relation, as regards position, between AB and DH?

How is the point D situated with reference to points A and B?

How is the point H situated with reference to the same points?

Then what must be the relation of AC and CB as regards magnitude?

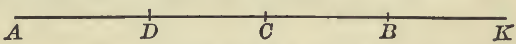
Suppose, now, DH be indefinitely extended, and any point in it selected at random. How will this point be situated with reference to the points A and B? Why?

What name, then, shall we give to the line NK? Why? Q.E.F.

727. **It is required to construct the perpendicular bisector of a given line.**

Sug. Employ method similar to the previous one.

728. **It is required to construct a perpendicular to a given straight line which shall pass through a given point in that line.**



Post. Let AK be the given line, and C the given point.

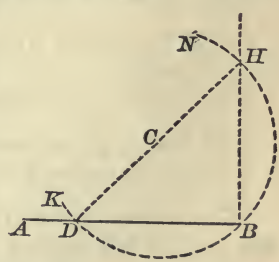
We are required to construct a perpendicular to AK passing through point C.

Sug. Lay off equal distances each side of C, as CD and CB; then use Problem 727.

729. **It is required to construct a perpendicular to a line at one extremity.**

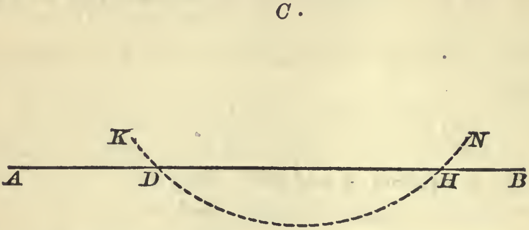
Sug. Extend the line; then use previous problem. In case the extension should not be possible or convenient, use the following:

Select any point at random, as C, making sure that it lies between A and B. Then, with CB as a radius, construct the circle or arc KBN. Through D, point where this circle intersects the given line, and C, draw the diameter DH. Join HB.



Then HB is the perpendicular required. Why? Q.E.F.

730. It is required to construct a perpendicular to a given line from a given point without the line.



Post. Let AB be the given line and C the given point.

Sug. With C as a center, construct a circle which shall intersect AB, as $KDHN$. Use Problem 727.

731. It is required to bisect a given arc.

Sug. Connect the extremities of the given arc; then use Problem 727, and for proof consult Theorems 310 and 299.

732. It is required to bisect a given angle.

Sug. Consult Theorem 95.

733. At a given point in a given line, it is required to construct an angle equal to a given angle, the given line forming one side of the angle.

Sug. Consult 293 and 292.

734. It is required to draw through a given point a line parallel to a given line.

Sug. Consult 115 or 108, and Problem 733.

735. Two angles of a triangle being given, it is required to construct the third angle.

Sug. Consult Theorems 126 and 72, and Problem 733.

736. Having given two sides of a triangle and their included angle, it is required to construct the triangle.

737. Having given two angles of a triangle and the side joining their vertices, it is required to construct the triangle.

When is this problem impossible?

738. Having given three sides of a triangle, it is required to construct the triangle.

When is this problem impossible?

739. It is required to construct the locus of the point which is a given distance from a given straight line.

740. It is required to construct a locus of the point which is a given distance from a given circumference.

741. It is required to construct the locus of a point which is equally distant from two given parallel lines.

742. It is required to construct the locus of the point which is equally distant from two non-parallel lines in the same plane.

743. It is required to construct the locus of the point which is equally distant from the circumferences of two equal circles.

744. Having given the hypothenuse of a right triangle, it is required to construct the locus of the vertex of the light angle.

Sug. Consult Theorem 394.

745. It is required to find in a given line, a point which is equally distant from two given points.

Sug. Consult Theorem 100.

746. It is required to find a point which is equidistant from three given points.

Sug. Join two pairs of points, then consult Theorem 100.

747. Through a given point without a given line, it is required to draw a line which shall make an angle with the given line equal to a given angle.

748. It is required to construct the triangle, having given the base, the vertical angle, and one of the other angles.

749. It is required to construct the triangle, having given two sides and an angle opposite one of them.

How many possible constructions?

750. It is required to find the center of a circle, having the circumference or an arc given.

751. It is required to construct the circumference, having given three points in it.

752. It is required to construct a circumference which shall pass through the vertices of a given triangle.

753. It is required to find the locus of the center of the circumference which shall pass through two given points.

754. With a given radius, it is required to construct the circle which shall pass through two given points.

755. It is required to construct the isosceles triangle, having given the base and the verticle angle.

756. It is required to construct a circumference which shall be a given distance from three given points.

757. It is required to construct a circle which shall have its center in a given straight line and circumference passing through two given points.

758. It is required to find a point which shall be equidistant from two given points, and at a given distance from a third given point.

759. It is required to construct the equilateral triangle, having given one side.

760. It is required to trisect a right angle.

761. It is required to find a point which shall be equally distant from two given points, and also equally distant from two given parallel lines.

762. It is required to find a point which shall be equally distant from two given points, and also equally distant from two given non-parallel lines in the same plane.

763. It is required to find a point which shall be equally distant from two given parallel lines, and also equally distant from two non-parallel lines in the same plane.

764. It is required to construct

i. an angle of 45°;	vi. an angle of 75°;
ii. an angle of 60°;	vii. an angle of 22° 30′;
iii. an angle of 30°;	viii. an angle of 52° 30′;
iv. an angle of 15°;	ix. an angle of 135°;
v. an angle of 105°;	x. an angle of 165°.

765. It is required to find a point in one side of a triangle which shall be equally distant from the other two sides.

766. It is required to find a point which shall be equally distant from two non-parallel lines in the same plane, and at a given distance from a given point.

767. It is required to construct the right triangle, having given the two legs.

768. It is required to construct the right triangle, having given the hypothenuse and one acute angle.

769. It is required to construct the right triangle, having given one leg and adjacent acute angle.

770. It is required to construct the right triangle, having given one leg and the acute angle opposite.

771. It is required to construct the right triangle, having given the hypothenuse and one leg.

772. It is required to construct a tangent to a given circle passing through a given point.

Case I. When the given point is the point of contact.

Case II. When the given point is outside the circle.

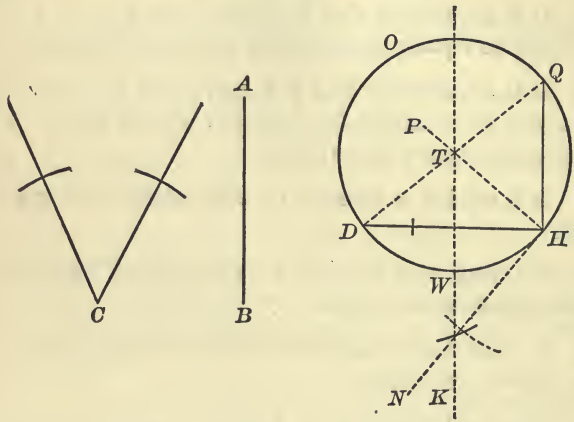
Sug. Connect the center of the given circle with the given exterior point. On this line as a diameter construct a circle, and join the points of its intersection of the given circle with extremities of the diameter.

773. It is required to construct the parallelogram, having given two sides and their included angle.

774. It is required to construct a circle within a given triangle, so that the sides of the triangle shall be tangents of the circle, *i.e.* to inscribe a circle.

Sug. Consult Theorems 146 and 149.

775. It is required to construct the circumference of a circle, having given a chord and angle made by the chord and a tangent to that circle.



Let AB be the given chord, and C the angle made by the chord and tangent. Then one extremity of the chord must be the point of contact.

Take $DH = AB$. Construct angle DHN equal to angle C, and draw $PH \perp$ to NH.

Then H is the point of contact, and NH the tangent.

Find the locus of the center of the circumference passing through D and H. Problem 753.

Hence T must be the center of the circle required.

If from any point in the arc DOH (call the point Q) lines be drawn to D and H, how will the angle Q compare in magnitude with the angle DHN?

What is formed by the lines DH, DQ, and HQ?

If we call DH the base, what would you call the angle Q?

What the point Q?

What might the arc DQH be named, then? Q.E.F.

776. Having given the base and vertical angle of a triangle, it is required to construct the locus of its vertex.

Sug. Consult the previous problem.

777. **It is required to construct the triangle, having given the base, the vertical angle, and the altitude.**

Sug. Use previous problem or 775.

778. **It is required to construct the triangle, having given the base, the vertical angle, and the median.**

OPTIONAL PROBLEMS FOR ADVANCED WORK.

779. It is required to construct the triangle, having given the base, vertical angle, and perpendicular from one extremity of the base to the opposite side.

780. It is required to construct the isosceles triangle, having given the altitude and one of the equal angles.

781. It is required to construct the chord in a given circle, having given the middle point of the chord.

782. It is required to construct a circle whose circumference shall pass through the vertices of a given rectangle.

783. It is required to construct a tangent to a given circle which shall be parallel to a given straight line.

784. It is required to construct a tangent to a given circle which shall be perpendicular to a given straight line.

785. It is required to construct a tangent to a given circle which shall make a given angle with a given straight line.

786. It is required to construct the isosceles triangle, having given the vertical angle, and a point in the base, in position.

787. It is required to construct the triangle, having given the altitude, the base, and an adjacent angle.

788. It is required to construct the triangle, having given the altitude, the base, and an adjacent side.

789. It is required to construct a rhombus, having given its base and altitude.

790. It is required to construct the triangle, having given the altitude and the sides which include the vertical angle.

791. It is required to construct the triangle, having given the altitude and angles adjacent to the base.

792. It is required to construct an isosceles triangle which shall have its vertical angle twice the sum of its other two angles.

793. It is required to construct the square, having given its diagonal.

794. It is required to construct the right triangle, having given the hypothenuse and perpendicular from the vertex of the right angle to the hypothenuse.

795. It is required to construct the locus of the center of the circle of given radius tangent to a given straight line.

796. It is required to construct the circle of given radius which shall be tangent to two given non-parallel lines.

797. It is required to construct the circle of given radius which shall be tangent to a given straight line, and whose center shall be in a given line not parallel to the former.

798. It is required to construct the circle which shall be tangent to a given line, and whose circumference shall pass through a given point.

799. It is required to find the locus of the center of the circle of a given radius which shall be tangent externally to a given circle.

800. It is required to construct the locus of the center of the circle of given radius which shall be tangent internally to a given circle.

801. It is required to construct a circle with a given radius which shall be tangent to a given line and a given circle.

802. It is required to construct a circle with a given radius which shall be tangent to two given circles.

803. It is required to construct a circle which shall cut three equal chords of given length from three given non-parallel lines.

804. It is required to construct in a given circle a chord of given length passing through a given point.

805. It is required to construct in a given circle a chord of given length and parallel to a given straight line.

806. It is required to construct a line of given length passing through a given point between two given parallel lines.

807. It is required to construct a line of given length between two given non-parallel lines, and which shall be parallel to a given line.

808. It is required to construct the right triangle, having given the hypothenuse and radius of the inscribed circle.

809. It is required to construct the right triangle, having given the radius of the inscribed circle and one acute angle.

810. It is required to construct the triangle, having given in position the middle points of its sides.

811. It is required to construct the triangle, having given the base, vertical angle, and radius of the circumscribed circle.

812. It is required to construct the isosceles triangle, having given the base and radius of the inscribed circle.

813. It is required to construct a straight line which shall pass through a given point and make equal angles with two given lines.

814. It is required to find a point in a given secant to a given circle such that the tangent to the circle from that point shall be of given length.

815. It is required to construct the right triangle, having given one leg and radius of the inscribed circle.

816. It is required to construct the right triangle, having given the median and altitude from the vertex of the right angle to the hypothenuse.

817. It is required to find the locus of the center of the chord which passes through a given point in a given circle.

818. It is required to construct the triangle, having given the base, vertical angle, and sum of the other two sides.

819. It is required to construct a circle which shall be tangent to two given lines and at given point of contact in one.

820. It is required to inscribe a circle in a given sector.

821. It is required to construct a common tangent to two given circles.

Sug. Make five cases according to the relative position of the circles. (See Theorem 322.)

822. It is required to inscribe a square in a given rhombus.

823. Having given two intersecting circles, it is required to draw a line through one of the points of intersection, so that the two intercepted chords shall be equal.

Sug. Join centers. Draw from point of intersection to middle of that line.

From centers draw radii parallel to latter line. Through point of intersection draw line perpendicular to these radii.

824. It is required to construct an equilateral triangle having its vertices in three given parallel lines.

825. It is required to construct a tangent to a given circle with two given parallel secants, so that the point of contact shall bisect the part between the secants.

826. It is required to construct three equal circles which shall be tangent to each other and also to a given circle externally.

827. It is required to construct three equal circles which shall be tangent to each other and also to a given circle internally.

828. It is required to construct three equal circles which shall be tangent to each other and also to the sides of an equilateral triangle.

829. It is required to construct a semicircle having its diameter in one of the sides of a given triangle and tangent to the other two sides.

830. It is required to construct a triangle, having given the radius of the inscribed circle and two sides.

831. It is required to find a point in a given line such that lines to that point from two given points without the line make equal angles with the line.

832. It is required to construct the triangle, having given the perimeter, altitude, and vertical angle.

PROBLEMS OF CONSTRUCTION, INVOLVING RATIO AND PROPORTION.

It is required,

833. **To divide a given line into any number of equal parts.**

Sug. From one extremity of the given line construct a line of indefinite length, making any convenient angle with the given line. Then, with any convenient unit of length assumed as a

unit of measure, beginning at the vertex, lay off on the indefinite line this unit as many times as it is required to divide the given line into parts. Then consult 422.

834. **To divide a given line into parts proportional to any number of given lines.**

Sug. Draw a line as in 833 and set off on this line segments equal to the given lines. Then consult 427.

835. **To construct a line that shall be a fourth proportional to three given lines.**

Sug. Consult Theorem 423.

836. **To construct a line that shall be a third proportional to two given lines.**

Sug. What must one of the given lines be if those two and one other are to form a proportion? Consider that in the construction.

837. **To construct a mean proportional between two given lines.**

Sug. Consult Theorem 449.

838. **Given a polygon and an homologous side of another similar polygon, to construct the latter.**

Sug. Consult Theorem 477.

839. To inscribe in a given circle a triangle similar to a given triangle.

840. To circumscribe about a given circle a triangle similar to a given triangle.

841. **To construct a square which shall be equivalent to the sum of two given squares.**

Sug. Consult Theorem 525.

842. **To construct a square which shall be equivalent to the sum of three or more given squares.**

Sug. Construct a square equivalent to two of the given squares, then one equivalent to that and one other of the given squares, and so on.

843. To construct a square which shall be equivalent to the difference of two given squares.

844. To construct a square equivalent to a given rectangle.

Sug. If x and y are the base and altitude of a rectangle, and n one side of an equivalent square, then

$$xy = n^2.$$

Whence $\quad x : n :: n : y \quad$ (see Theorem 357)

or n is a mean proportional between x and y.

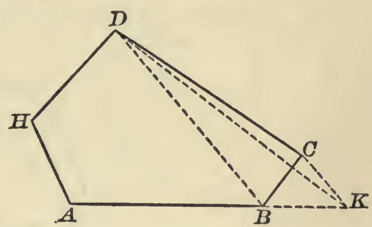
Hence consult Problem 837.

845. To construct a square which shall be equivalent to a given parallelogram.

846. To construct a square which shall be equivalent to a given triangle.

847. To construct a triangle which shall be equivalent to a given polygon of more than three sides.

Post. Let $ABCDH$ be a polygon of n sides.



Extend one of the sides as AB, and construct the diagonal DB. Through vertex C draw CK parallel to DB and join DK.

Considering DB the common base of the two triangles DCB and DKB, what is the relation between the areas of those two triangles? Why?

How does the area of the polygon $DKAH$ then compare with that of $DCBAH$? Why?

How many sides has the former as compared with the latter?

Proceed in the same way with the polygon $DKAH$.

848. To construct a square equivalent to any given polygon.

Sug. Use Problems 847 and 846.

849. To construct a square equivalent to the sum of any number of given polygons.

850. To construct a rectangle which shall be equivalent to a given square, and the sum of whose base and altitude shall be equal to a given line.

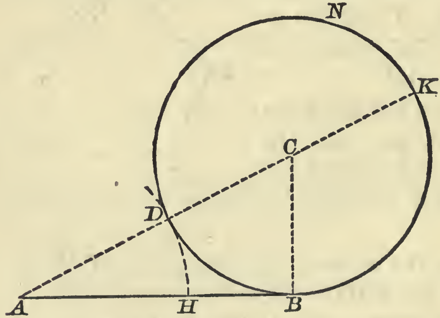
Sug. Upon the given line as a diameter construct a circle. Construct a line parallel to the diameter, distant from it one side of the given square. Then consult Theorem 449.

851. To construct a rectangle which shall be equivalent to a given square, and the difference of whose base and altitude shall be equal to a given line.

Sug. Proceed as in 850, then at extremity of the diameter construct a tangent equal to one side of a given square, and from other extremity of this tangent construct a secant through center of circle. (Consult Theorem 452.)

852. To construct a right triangle which shall be equivalent to a given triangle and its hypothenuse equal to a given line.

853. To divide a given line into extreme and mean ratio. (See 348.)



Post. Let AB be the given line. At one extremity erect a perpendicular CB equal to one half AB. With C as a center and CB as a radius construct the circle DBN. Construct the secant AK passing through the center C. With A as a center

and radius AD, construct the arc DH. Then the line AB will be divided in extreme and mean ratio at H.

Dem. Since $CB = \frac{AB}{2}$, $DK = AB$.

$AD : AB :: AB : AK$ Why?

$AD : AB - AD :: AB : AK - AB$ Why?

$AH : AB - AH :: AB : AK - DK$ Why?

$AH : HB :: AB : AD$ Why?

$AH : HB :: AB : AH$ Why?

or $HB : AH :: AH : AB$ Why?

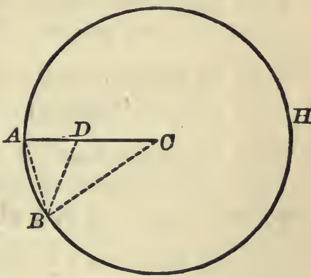
Hence the line AB is divided in extreme and mean ratio.

Q.E.F.

854. To inscribe a regular decagon in a given circle.

Post. Let ABH be the given circle, and AC one of its radii. Consult Problem 853.

Cons. With A as a center and DC as a radius, construct chord AB. Join BC and BD.



Dem.

$AC : AB :: AB : AD$ Why?

Hence the two $\triangle$ ABD and ABC are how related? (See Theorem 441.)

What kind of a triangle is ACB?

What kind of a triangle must ABD be, then?

What relation, then, between AB and DB? What between BD and DC?

What relation between the $\angle$ CAB and CBA? Between the $\angle$ DAB and BDA? Between the $\angle$ BDA and CBA? Why? Between the $\angle$ DBC and DCB?

What relation does the $\angle$ BDA bear to the sum of DCB and DBC? Why?

What relation does $\angle DAB$ bear to $\angle C$, then?

Hence, what relation does $\angle ABC$ bear to $\angle C$?

Compare now $\angle DAB + \angle ABC$ with the $\angle C$, and finally compare $\angle DAB + \angle ABC + \angle C$ with the $\angle C$.

If the pupil has answered the above questions correctly, he will have now the equation

$$\angle DAB + \angle ABC + \angle C = 5 \angle C.$$

What is the value of the first member of the above equation? Why?

Then $$5 \angle C = 2 \text{ rt. } \angle s$$

or $$10 \angle C = 4 \text{ rt. } \angle s$$

Whence $$\angle C = \frac{1}{10} \text{ of } 4 \text{ rt. } \angle s.$$

Hence the arc AB is what part of the circumference?

$\therefore$ AB is the side of a regular inscribed decagon. Q.E.F.

855. To inscribe a square in a given circle.

Sug. Consult Theorem 639.

856. To inscribe a regular hexagon in a given circle.

Sug. Consult Theorem 471.

857. To inscribe a regular pentedecagon in a given circle.

Sug. Find the difference between the central angle of the regular decagon and that of a regular hexagon.

858. To inscribe in a given circle

i. a regular trigon;
ii. a regular pentagon;
iii. a regular octagon;
iv. a regular dodecagon;
v. a regular polygon of sixteen sides;
vi. a regular polygon of twenty sides.

859. To circumscribe about a given circle all the above-mentioned regular polygons.

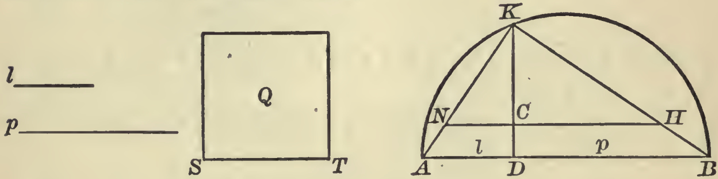
860. To inscribe in a given circle a regular polygon similar to a given regular polygon.

Sug. Construct a central angle equal to that of the given polygon, etc.

861. To circumscribe a circle about any given regular polygon.

862. Upon a given line as a base, to construct a rectangle equivalent to a given rectangle.

863. To construct a square whose ratio to a given square shall be the same as that of two given lines.



Post. Let Q be the given square, and l and p the two given lines.

We are required to construct a square (Q') so that

$$\text{Area } Q : \text{area } Q' :: l : p.$$

Cons. Draw $AB = l + p$.

On this as a diameter construct a semicircle, and at D erect the perpendicular DK.

Join AK and BK. Make KN equal to one side of the given square, as ST.

Draw NH parallel to AB. Then KH is the side of the required square.

Dem. $\overline{AK}^2 : \overline{BK}^2 :: AD : DB$ (?)

or $\overline{AK}^2 : \overline{BK}^2 :: l : p$ (?)

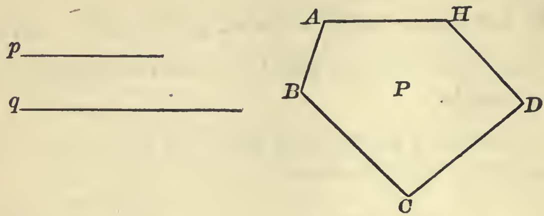
Again, $NK : HK :: AK : BK$ (?)

Hence $\overline{NK}^2 : \overline{HK}^2 :: \overline{AK}^2 : \overline{BK}^2$ (?)

$\therefore \overline{NK}^2 : \overline{HK}^2 :: l : p$ (?)

Hence the square constructed on HK as a side is the required square. Q.E.F.

864. **To construct a polygon similar to a given polygon, the ratio of whose areas shall be that of two given lines.**



Post. Let $ABCDH$ be the given polygon (P), and p and q the two given lines.

We are required to construct a polygon (P') similar to P, so that

$$\text{Area } P : \text{area } P' :: p : q.$$

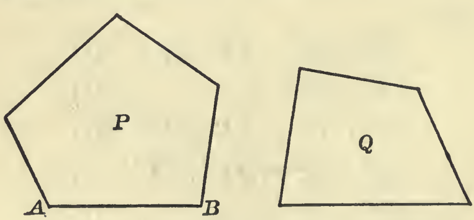
Cons. Upon any side of the polygon P, as AB, construct a square.

Then, by the previous problem, find the side of a square whose ratio to that of the square on AB shall be that of the two lines p and q.

Upon this line construct a polygon similar to polygon P. This will be the polygon required.

Dem. This will be left for the pupil.

865. **To construct a polygon similar to one of two given polygons and equivalent to the other.**



Post. Let P and Q be the two given polygons. We are required to construct a polygon similar to P and equivalent to Q.

Cons. Construct squares equivalent to each of the polygons P and Q.

Then find a fourth proportional to the sides of these squares and any side of the polygon P selected at random, as AB. Upon this fourth proportional as an homologous side construct a polygon similar to P. Then this polygon will be similar to P and equivalent to Q, and is therefore the polygon required.

Dem. This is also left for the pupil.

866. To construct upon a given line, as one side,

i. a regular trigon;
ii. a regular tetragon;
iii. a regular hexagon;
iv. a regular octagon;
v. a regular pentagon;
vi. a regular decagon;
vii. a regular dodecagon;
viii. a regular pentedecagon.

867. To construct a regular hexagon, given one of its shorter diagonals.

868. To construct a regular pentagon, given one of its diagonals.

869. To construct a circle equivalent to the sum of two given circles.

870. To construct a circumference equal to the sum of two given circumferences.

871. To divide a given circle by a concentric circumference into two equivalent parts.

MISCELLANEOUS PROBLEMS FOR ADVANCE WORK.

I. Triangles.

It is required to construct the triangle, having given

872. Its base, vertical angle, and difference of the other two sides.

873. Its base, vertical angle, and a square which is equal to the sum of the squares upon the other two sides.

874. Its base, vertical angle, and a square which is equivalent to the difference of the squares upon the other two sides.

875. Its base, vertical angle, and sum of its altitude and the two remaining sides.

876. Its base, vertical angle, and the sum of its altitude and difference of the other two sides.

877. Its base, vertical angle, and difference between its altitude and sum of its other two sides.

878. Its base, vertical angle, and difference between its altitude and difference of the other two sides.

879. Its base, vertical angle, and ratio of its altitude to the sum of its other two sides.

880. Its base, vertical angle, and ratio of its altitude to the difference of its other two sides.

881. Its base, altitude, and sum of its other two sides.

882. Its base, altitude, and difference of its other two sides.

883. Its base, altitude, and ratio of the other two sides.

884. Its base, altitude, and a square equivalent to the rectangle of the other two sides.

885. Its base, altitude, and a square which is equivalent to the sum of the squares upon the other two sides.

886. Its base, altitude, and a square which is equivalent to the difference of the squares upon the other two sides.

887. Its vertical angle, sum of base and altitude, and sum of the other two sides.

888. Its vertical angle, sum of base and altitude, and difference of its other two sides.

889. Its vertical angle, sum of base and altitude, and ratio of the other two sides.

890. Its vertical angle, sum of base and altitude, and a square equivalent to the rectangle of its other two sides.

891. Its vertical angle, sum of base and altitude, and sum of the three sides.

892. Its vertical angle, sum of base and altitude, and difference between the base and sum of the other two sides.

893. Its vertical angle, sum of base and altitude, and difference between the base and difference of its other two sides.

894. Its vertical angle, sum of base and altitude, and the ratio of the base to the sum of the other two sides.

895. Its vertical angle, sum of base and altitude, and the ratio of the base to the difference of its other two sides.

896. Its vertical angle, altitude, and radius of the circumscribed circle.

897. Its vertical angle, radius of the inscribed circle, and perimeter.

898. Its vertical angle, radius of the inscribed circle, and ratio of the sides including the vertical angle.

899. Its vertical angle, radius of the inscribed circle, and a square equivalent to the rectangle of the sum of the two sides including the vertical angle and the base.

900. Its vertical angle, radius of the inscribed circle, and a square whose area is equal to the difference between the sum of the squares of the sides including the vertical angle and the square of the base.

901. Its base, median, and sum of the other two sides.

902. Its base, median, and difference of the other two sides.

903. Its three altitudes.

904. Its three medians.

905. Two sides, and difference of the angles opposite them.

906. Its vertical angle, difference of the angles at the base, and difference of the other two sides.

907. Difference of the angles at the base, difference of the segments of the base made by the altitude, and sum of the other two sides.

908. It is required to construct the equilateral triangle, having given the three distances from the vertices to a common point.

II. Quadrilaterals.

909. It is required to construct a square, having given

i. The sum of its diagonal and side.

ii. The difference of its diagonal and side.

910. It is required to construct a rectangle, having given

i. The sum of two adjacent sides and its diagonal.

ii. The difference of two adjacent sides and its diagonal.

iii. One side, and sum of diagonal and adjacent side.

iv. One side, and difference of diagonal and adjacent side.

911. It is required to construct the rhombus, having given

i. Its altitude and lesser angle.

ii. Its side and sum of its diagonals.

iii. Its side and difference of its diagonals.

iv. Its lesser angle and sum of its diagonals.

v. Its lesser angle and difference between its longer diagonal and altitude.

912. It is required to construct the rhomboid, having given

i. The longer side, sum of its diagonals, and larger angle made by the diagonals.

ii. Its lesser angle, longer diagonal, and sum of two adjacent sides.

iii. Its lesser angle, longer side, and sum of its altitude and lesser side.

iv. Its lesser angle, shorter side, and difference of its longer diagonal and longer side.

913. It is required to construct an isosceles trapezoid, having given

i. One leg, diagonal, and longer base.
ii. Its longer base, diagonal, and lesser angle.
iii. Its diagonal, altitude, and leg.
iv. Its longer base, lesser angle, and sum of altitude and leg.

914. It is required to construct the trapezoid, having given

i. Its longer base, lesser angle (*i.e.* angle formed by longer base and a leg), and its diagonal.
ii. Its longer base, one leg, lesser angle, and altitude.
iii. Sum of its bases, the two legs, and lesser angle.
iv. Difference of its bases, the two legs, and angle formed by its diagonals.

III. Circles.

915. Having given a circle, it is required to construct

i. Three equal circles, tangent to the given circle externally and tangent to each other.
ii. Three equal circles, tangent to the given circle internally and tangent to each other.
iii. Four equal circles, as in **i** and **ii**.
iv. Five equal circles, as in **i** and **ii**.
v. Six equal circles, as in **i** and **ii**.
vi. A circle tangent to three given circles.

IV. Transformation of Figures.

916. To transform a given triangle into an equivalent isosceles one having the same base.

917. To transform a given isosceles triangle into an equivalent equilateral one.

918. To transform a given triangle into an equivalent equilateral one.

919. To transform a given triangle into another equivalent triangle whose base and altitude shall be equal.

920. To transform a given triangle into another equivalent triangle, and similar to a given triangle.

921. To transform a given triangle into a triangle with one angle unchanged and its opposite side parallel to a given line.

922. To transform a given triangle into an equivalent triangle with a given perimeter.

923. To transform a triangle into a trapezoid, one of whose bases shall be the base of the triangle, and one of its adjacent angles one of the basal angles of the triangle.

924. To transform a given triangle into a right triangle with given perimeter.

925. To transform a given triangle into a parallelogram with given base and altitude.

926. To transform a parallelogram into a parallelogram with a given side.

927. To transform a parallelogram into a parallelogram having a given angle.

928. To transform a parallelogram into a parallelogram with given altitude.

929. To transform a square into

- **i.** A right triangle.
- **ii.** An isosceles triangle.
- **iii.** An equilateral triangle.
- **iv.** A rectangle with given side.
- **v.** A rectangle with given perimeter.
- **vi.** A rectangle with given difference of sides.
- **vii.** A rectangle with given diagonal.

930. To transform a rectangle into

i. A square.
ii. An isosceles triangle.
iii. An equilateral triangle.
iv. A rectangle with given side.
v. A rectangle with given perimeter.
vi. A rectangle with given difference of sides.
vii. A rectangle with given diagonal.

931. It is required to construct a parallelogram equivalent to the

i. Sum of two given parallelograms of equal altitudes.
ii. Difference of two given parallelograms of equal altitudes.
iii. Sum of two given parallelograms of equal bases.
iv. Difference of two given parallelograms of equal bases.
v. Sum of two given parallelograms.
vi. Difference of two given parallelograms.

932. It is required to transform a given parallelogram into

i. A triangle.
ii. An isosceles triangle.
iii. A right triangle.
iv. An equilateral triangle.
v. A square.
vi. A rhombus having for a diagonal one side of the parallelogram.
vii. A rhombus having a given diagonal.
viii. A rhombus having a given side.
ix. A rhombus having a given altitude.
x. A parallelogram having a given side and diagonal.

933. To transform a trapezoid into

i. A triangle.
ii. A square.
iii. A parallelogram having for one base the longer base of the trapezoid.
iv. An isosceles trapezoid.

934. To transform a trapezium into

i. A triangle.

ii. An isosceles triangle with given base.

iii. A parallelogram.

iv. A trapezoid with one side and the two adjacent angles unchanged.

V. Division of Figures.

935. It is required to divide a given triangle into any number of equivalent parts by lines drawn from one vertex.

936. It is required to divide a given triangle into any number of equivalent parts by lines drawn from any point in its perimeter.

937. It is required to divide a given triangle into any number of equivalent parts by lines drawn from any point in the triangle.

938. It is required to divide a given triangle into any number of equivalent parts by lines parallel to one side.

939. It is required to divide a given triangle into any number of parts, whose areas shall be in a given ratio, by lines drawn from one vertex.

940. It is required to divide a given triangle into any number of parts, whose areas shall be in a given ratio, by lines drawn from any point in the perimeter.

941. It is required to divide a given triangle into any number of parts, whose areas shall be in a given ratio, by lines drawn from any point in the triangle.

942. It is required to divide a given triangle into any number of parts, whose areas shall be in a given ratio, by lines drawn parallel to one side.

943. It is required to divide a triangle in the ratio of m to n by a line perpendicular to the base.

944. It is required to divide a given parallelogram into any number of equal parts by lines drawn parallel to one pair of sides.

945. It is required to divide a given parallelogram into any number of parts, whose areas shall be in a given ratio, by lines parallel to one pair of sides.

946. It is required to divide a given parallelogram into two equivalent parts by a line drawn from any point in the perimeter.

947. It is required to divide a given parallelogram into two equivalent parts by a line drawn through any point in the parallelogram.

948. It is required to divide a given parallelogram into two parts, whose areas shall be in a given ratio, by a line drawn from one vertex.

949. It is required to divide a given parallelogram into two parts, whose areas shall be in a given ratio, by a line drawn from any point in the perimeter.

950. It is required to divide a given parallelogram into two parts, whose areas shall be in a given ratio, by a line drawn from any point in the parallelogram.

951. It is required to divide a parallelogram into two equivalent parts by a line drawn parallel to a given line.

952. It is required to divide a parallelogram into two parts, whose areas shall be in a given ratio, by a line drawn parallel to a given line.

953. It is required to divide a parallelogram into any number of equivalent parts by lines drawn from either vertex.

954. It is required to divide a parallelogram into any number of equivalent parts by lines drawn from any point in its perimeter.

955. It is required to divide a parallelogram into any number of equivalent parts by lines drawn from any point in the parallelogram.

956. It is required to divide a parallelogram into any number of equivalent parts by lines parallel to a given line.

957. It is required to divide a parallelogram into any number of parts, whose areas shall be in a given ratio, by lines parallel to a given line.

958. It is required to divide a trapezoid into two equivalent parts by a line drawn

i. Parallel to the bases.
ii. Perpendicular to the bases.
iii. Parallel to one of the legs.
iv. Through one of its vertices.
v. Through a given point in one of its bases.
vi. Through any point in its perimeter.
vii. Through any point in the trapezoid.
viii. Parallel to a given line.

959. It is required to divide a trapezoid into any number of equivalent parts by lines drawn

i. Parallel to the bases.
ii. Perpendicular to the bases.
iii. Parallel to one of its legs.
iv. Through one of its vertices.
v. Through any point in one of its bases.
vi. Through any point in its perimeter.
vii. Through any point in the trapezoid.
viii. Parallel to a given line.

960. It is required to divide a given trapezoid into any number of parts whose areas shall be in a given ratio by lines drawn

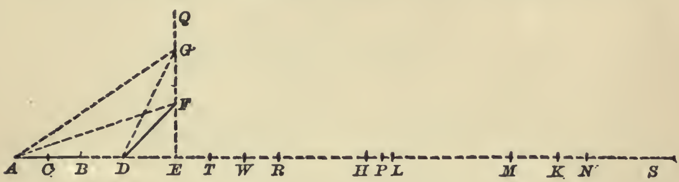
i. Parallel to the bases.
ii. Perpendicular to the bases.
iii. Parallel to one of the legs.
iv. Through either vertex.
v. Through any point in one of the bases.
vi. Through any point in its perimeter.
vii. Through any point in the trapezoid.
viii. Parallel to a given line.

961. It is required to divide a trapezium into two equivalent parts by a line drawn from either vertex.

962. It is required to divide a trapezium into two equivalent parts by a line drawn from any point in its perimeter.

963. Given the diameter of a circle, it is required to construct a straight line equal to the circumference in length.

From 654 it is evident that it can only be approximated.



Let AB be the given diameter and C its middle point. Extend AB indefinitely as AS. Make BD and DE each equal to AB. At E erect the perpendicular EQ, and on it make EF and FG each equal to AB. Join AG, AF, DG, and DF. Lay off EH and HK each equal to AG, and from K lay off KL equal to AF. Again, from L make LM equal to DG, and MN equal DF. Bisect EN at P; bisect EP at R; and then trisect ER at T and W. Then CT will be the required line approxi-

mately equal to the circumference of the circle whose diameter is AB; for, calling the diameter unity,

$$CE = 2\tfrac{1}{2},$$

$$EL = 2\,CH - KL = 2\sqrt{13} - \sqrt{10},$$

$$LM = \sqrt{5},$$

$$MN = \sqrt{2}.$$

$$\therefore\ EN = 2\sqrt{13} - \sqrt{10} + \sqrt{5} + \sqrt{2},$$

and $$ET = \tfrac{1}{12}(2\sqrt{13} - \sqrt{10} + \sqrt{5} + \sqrt{2}),$$

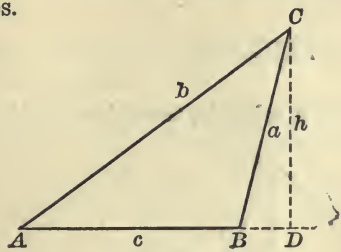
and therefore

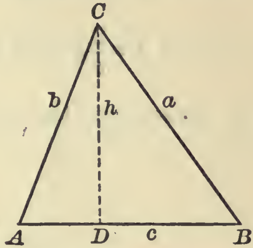
$$CT = 2\tfrac{1}{2} + \tfrac{1}{12}(2\sqrt{13} - \sqrt{10} + \sqrt{5} + \sqrt{2}) = 3.1415922+.$$

Q.E.F.

SOLUTIONS BY ALGEBRA.

964. To compute the altitude of a triangle in terms of its sides.





Let $\angle A$ be an acute angle. $\therefore$ the altitude, designated by h, may fall either without or within the triangle. In either case

$$a^2 = b^2 + c^2 - 2\,c \cdot AD.$$

$$\therefore\ AD = \frac{b^2 + c^2 - a^2}{2\,c}.$$

$$\overline{AD}^2 + h^2 = b^2.$$

$$\therefore\ h^2 = b^2 - \overline{AD}^2.$$

$$\therefore\ h^2 = b^2 - \left(\frac{b^2 + c^2 - a^2}{2\,c}\right)^2.$$

$$\therefore\ h^2 = \frac{4\,b^2c^2 - (b^2 + c^2 - a^2)^2}{4\,c^2}.$$

$$\therefore\ h^2 = \frac{(2\,bc + b^2 + c^2 - a^2)(2\,bc - b^2 - c^2 + a^2)}{4\,c^2}.$$

$$\therefore\ h^2 = \frac{(b + c + a)(b + c - a)(a + b - c)(a - b + c)}{4\,c^2}.$$

Now let one half the sum of the sides be designated by s.

$$\therefore\ a + b + c = 2\,s,$$

$$b + c - a = 2(s - a),$$

$$a + b - c = 2(s - c),$$

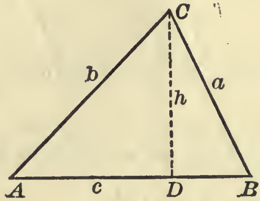
$$a - b + c = 2(s - b).$$

$$\therefore\ h^2 = \frac{2s \cdot 2(s-a)\,2(s-b)\cdot 2(s-c)}{4c^2}.$$

$$\therefore\ h^2 = \frac{16\,s(s-a)(s-b)(s-c)}{4c^2}.$$

$$\therefore\ h = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}.$$

965. To compute the area of a triangle, the sides only being given.

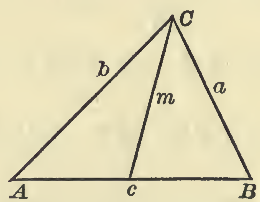


$$\text{Area } ABC = \frac{hc}{2} = \frac{c}{2}\cdot h;$$

$$h = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}.$$

$$\therefore\ \text{Area } ABC = \sqrt{s(s-a)(s-b)(s-c)}.$$

966. To compute the medians of a triangle in terms of its sides.



Designate a median by m.

$$\therefore\ a^2 + b^2 = 2m^2 + 2\left(\frac{c}{2}\right)^2.$$

$$\therefore\ a^2 + b^2 = 2m^2 + \frac{c^2}{2}.$$

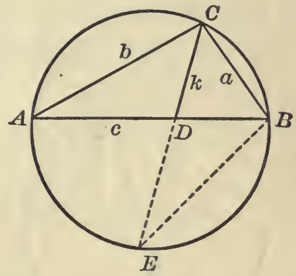
$$\therefore\ 4m^2 = 2(a^2+b^2) - c^2,$$

$$2m = \sqrt{2(a^2+b^2)-c^2},$$

$$m = \tfrac{1}{2}\sqrt{2(a^2+b^2)-c^2}.$$

967. To compute the bisector of an angle of a triangle in terms of its sides.

Circumscribe a circle about the triangle ABC. Extend CD, the bisector of the $\angle ACB$, to meet the circumference in E, and draw BE.



$$k^2 + AD \cdot BD = ab,$$

$$\therefore\ k^2 = ab - AD \cdot BD.$$

$$AD : BD :: b : a.$$

$$\therefore\ AD : b :: BD : a.$$

$$\therefore\ AD + BD : AD :: b + a : b.$$

$$\therefore\ AD + BD : a + b :: AD : b.$$

$$\therefore\ AD + BD : a + b :: AD : b :: BD : a;$$

or
$$\frac{c}{a+b} = \frac{AD}{b} = \frac{BD}{a}.$$

$$\therefore\ AD = \frac{bc}{a+b} \text{ and } BD = \frac{ac}{a+b}.$$

$$\therefore\ k^2 = ab - \frac{abc^2}{(a+b)^2}.$$

$$\therefore\ k^2 = ab\left(1 - \frac{c^2}{(a+b)^2}\right).$$

$$\therefore\ k^2 = \frac{ab\left[(a+b)^2 - c^2\right]}{(a+b)^2}.$$

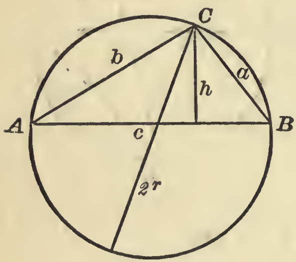
$$\therefore\ k^2 = \frac{ab(a+b+c)(a+b-c)}{(a+b)^2}.$$

$$\therefore\ k^2 = \frac{ab \cdot 2s\,2\,(s-c)}{(a+b)^2}.$$

$$\therefore\ k^2 = \frac{4\,abs\,(s-c)}{(a+b)^2}.$$

$$\therefore\ k = \frac{2}{a+b}\sqrt{abs\,(s-c)}.$$

968. To compute the area of a triangle in terms of its sides and radius of the circumscribing circle.



$$b : 2\,r :: h : a,$$

$$\therefore\ ab = 2\,rh.$$

$$\therefore\ abc = 2\,rch.$$

$$\text{Area } ABC = \frac{ch}{2},$$

$$ch = \frac{abc}{2\,r},$$

$$\therefore\ \text{Area } ABC = \frac{abc}{4\,r}.$$

969. To compute the radius of the circumscribing circle of a triangle in terms of its sides.

Sug. Use previous diagram.

$$ab = 2\,rh.$$

$$\therefore\ r = \frac{ab}{2\,h} = \frac{ab}{2} \div h,$$

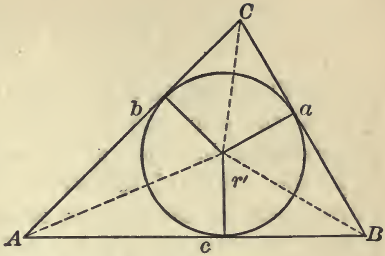
$$h = \frac{2}{c}\sqrt{s\,(s-a)\,(s-b)\,(s-c)},$$

$$h = \frac{2\sqrt{s\,(s-a)\,(s-b)\,(s-c)}}{c}.$$

$$\therefore\ \frac{ab}{2} \times \frac{c}{2\sqrt{s\,(s-a)\,(s-b)\,(s-c)}},$$

$$\therefore\ r = \frac{abc}{4\sqrt{s\,(s-a)\,(s-b)\,(s-c)}}.$$

970. To compute the radius of the inscribed circle in terms of the sides of the triangle.



$$\text{Area } ABC = \frac{cr'}{2} + \frac{ar'}{2} + \frac{br'}{2} = \frac{(a+b+c)r'}{2} = \frac{2sr'}{2} = sr'.$$

$$r' = \frac{\text{Area } ABC}{s} = \frac{1}{s}\sqrt{s(s-a)(s-b)(s-c)}.$$

971. The sides of a triangle are 480 feet, 600 feet, and 720 feet. Compute

i. its area;
ii. the shortest altitude;
iii. the longest median;
iv. the three angle bisectors;
v. radius of circumscribed circle;
vi. radius of inscribed circle.

972. The sides of a triangle are 36, 40, and 44 meters. Compute the areas in ares of the two triangles formed by the bisector of the angle between the last two sides.

973. Two sides of a parallelogram are 15 and 22 inches, and one diagonal is 18 inches. Compute the other diagonal and the area.

974. The sides of a triangle are 42, 35, and 28 meters. Compute the area of the triangle formed by the median and angle bisector drawn between the first two sides.

975. The sides of a triangle are 54, 72, and 80 feet. Compute
i. its area; ii. the area of the circumscribed circle;
iii. the area of the inscribed circle.

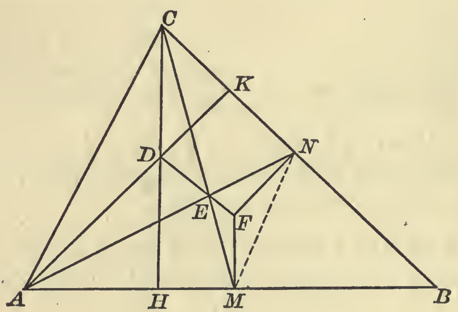
976. A field in form of a trapezium has one right angle ($\angle A$), and its sides are $AB = 58$ rods, $BC = 55$ rods, $CD = 37$ rods, and $AD = 32$ rods. Compute the area of the field.

SPECIAL THEOREMS.

977. The point of concurrence of the altitudes of a triangle is called the *orthocenter,* and the point of concurrence of the perpendicular bisectors of the sides is called the *circumcenter,* being the center of the circumscribed circle; the point of concurrence of the medians is called the *midcenter,* and the point of concurrence of the angle bisectors is called the *incenter.*

A complete quadrilateral is the figure formed by four straight lines intersecting one another in six points.

978. In every triangle the midcenter, the orthocenter, and the circumcenter are collinear; and the line joining the first two is twice that joining the last two.



Post. Let D be the orthocenter, E the midcenter, and F the circumcenter of the triangle ABC.

To Prove. DE and EF are one line, and $DE = 2\,EF$.

Cons. Draw NM.

Dem. NF is $\parallel$ to AD, (?) FM is $\parallel$ to CD, (?) NM is $\parallel$ to AC. (?)

$$\therefore \triangle ACD \text{ is similar to } \triangle FNM. \quad (?)$$

$2\,NM = AC$. (?) $\qquad \therefore \dfrac{NM}{AC} = \dfrac{1}{2}$. (?)

$NF : AD :: NM : AC.$ (?) $\therefore \frac{NF}{AD} = \frac{NM}{AC} = \frac{1}{2}.$ (?)

$\frac{NE}{AE} = \frac{1}{2}.$ (?) $\therefore \frac{NE}{AE} = \frac{NF}{AD}.$ (?)

$\therefore NE : AE :: NF : AD.$ $\angle DAE = \angle ENF.$ (?)

$\therefore \triangle DAE$ is similar to $\triangle ENF.$ (?)

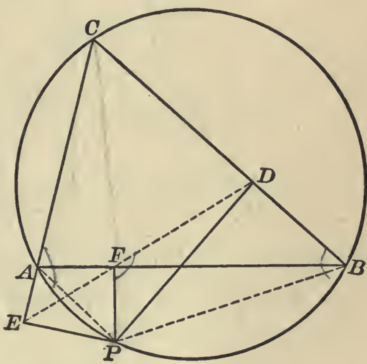
$\therefore \angle DEA = \angle NEF.$ (?)

$\therefore DE$ and EF are in the same line.

$\therefore NE : AE :: EF : DE.$ (?) $\therefore \frac{EF}{DE} = \frac{NE}{AE} = \frac{1}{2}.$ (?)

Q.E.D.

979. If a triangle be inscribed in a circle, and lines perpendicular to the sides be drawn from any point in the circumference, the feet of these perpendiculars are collinear.



Post. Let the ⊥s be PE, PF, and PD.

To Prove. E, F, and D are collinear.

Cons. Draw EF, FD, AP, and PB.

Dem. If a circle be described with AP a diameter, its circumference will pass through E and F.

Similarly, if a circumference be described with PB a diameter, it will pass through F and D. (?)

$\therefore \angle EAP = \angle EFP;$ (?)

$\angle CAP$ is sup. to $\angle EAP;$ (?)

$\angle CAP$ is sup. to $\angle CBP.$ (?)

$\therefore \angle EAP = \angle CBP = \angle EFP$; (?)

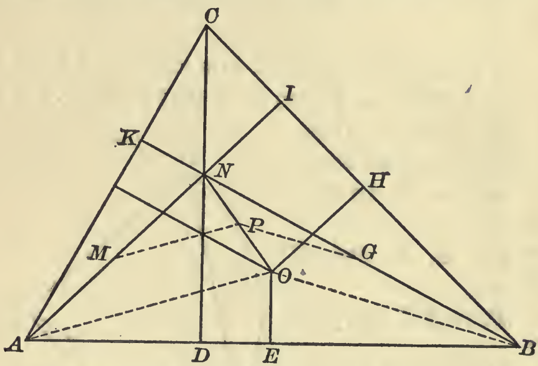
$\angle CBP$ is sup. to $\angle PFD$. (?)

$\therefore \angle EFP$ is sup. to $\angle PFD$. (?)

$\therefore$ EF and FD form one line.

$\therefore$ E, F, and D are collinear. Q.E.D.

980. If the middle point of the line joining the orthocenters and circumcenters be used as a center and a circle described with a radius equal to one half the radius of the circumscribed circle, the circumference will bisect the segments of the altitudes between the orthocenter and the vertices.



Post. Let N be the orthocenter and O the circumcenter of the $\triangle ABC$. Let M, P, and G be the mid-points of AN, NO, and NB respectively.

To Prove. $PM = PG$, etc.

Cons. Draw MP, PG, OA, and OB.

Dem. $MP = \frac{1}{2} AO$ and $PG = \frac{1}{2} OB$. (?)

$AO = OB$. (?)

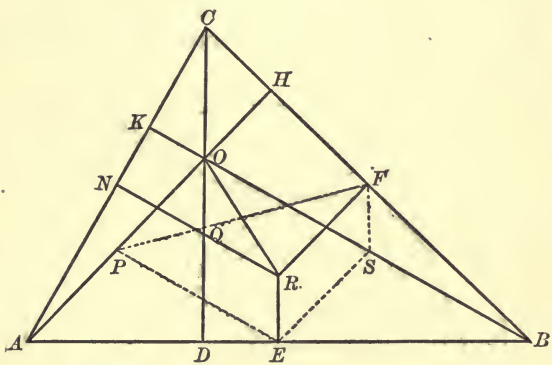
$\therefore MP = PG$. (?)

∴ if a circle be described with P as a center and radius PM, the circumference will pass through G, and therefore bisect NA and NB.

In a similar manner this circumference will bisect NC.

Q.E.D.

981. The circumference which bisects the segments of the altitudes of a triangle between the orthocenter and the vertices, bisects also the sides and passes through the feet of the altitudes. (This is called the nine-point circle.)



Post. Let OR be the line joining the orthocenter and circumcenter and Q its center.

To Prove. That the circumference described with Q as a center and $\frac{AR}{2}$ as a radius will pass through points D, E, F, H, K, and N.

Cons. Draw $FS \parallel CO$, and $EP \parallel BO$.

Join FP and ES.

Dem. FS is $\parallel$ to RE. (?)

ES is $\parallel$ to PO. (?)

RF is $\parallel$ to PO. (?)

∴ ES is $\parallel$ to RF. (?)

$\therefore \quad RF = PO.$ (?)

$\therefore \triangle POQ = \triangle FRQ.$ (?)

$\therefore$ *PF* bisects *OR*, *i.e.* passes through point *Q*.

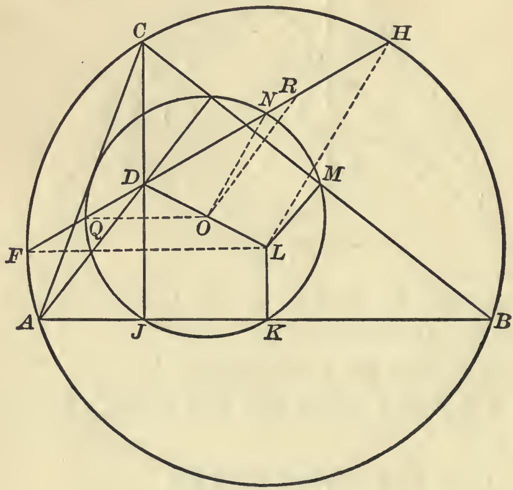
$\therefore PQ = QF.$ (Homol. sides of equal $\triangle$s.)

The $\perp$ bisectors of *HF*, *KN*, and *DE* pass through *Q* (422).

$\therefore$ *Q* is equidistant from points *D*, *E*, *F*, *H*, *K*, *N*, and the mid-points of *OA*, *OB*, and *OC*.

$\therefore$ the circumference which passes through one will pass through all. Q.E.D.

982. The circumference of the nine-point circle bisects every line drawn from the orthocenter to the circumference of the circumscribed circle.



Post. Let *O* be the center of the nine-point circle, and *FH* any chord of the circumscribed circle passing through the orthocenter *D*.

To Prove. $DN = NH$ and $DQ = FQ$.

Cons. Draw *OQ*, *ON*, *LH*, and *LF*, and draw *OR* to the mid-point of *DH*.

Dem. $OQ = \frac{1}{2} FL$ (?) and $ON = \frac{1}{2} LH$. (?)

$\therefore\ OQ = ON$. (?)

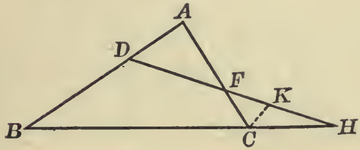
OR is $\parallel$ to LH, and $OR = \frac{1}{2} LH$. (?)

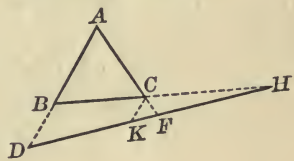
$\therefore\ OR = ON$. (?)

$\therefore\ OR$ coincides with ON. (?)

$\therefore\ N$ is the center of DH. (?) Q.E.D.

983. If a transversal cuts the sides of a triangle, produced if necessary, the product of three non-adjacent segments of the sides is equal to the product of the other three segments.





Post. Let ABC be a triangle, and DFH a transversal intersecting the sides.

To Prove. $BH \cdot FC \cdot AD = HC \cdot AF \cdot DB$.

Cons. Draw $CK \parallel$ to DB.

Dem. KC is $\parallel$ to DB. (?)

$\therefore\ BH : CH :: DB : CK$, (?)

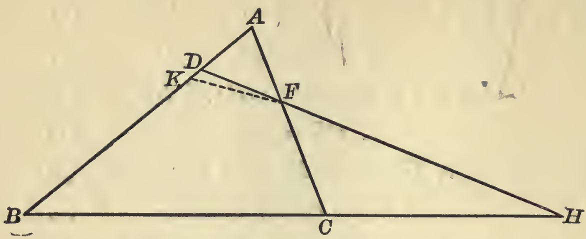
and $FC : AF :: CK : AD$. (?)

$\therefore\ BH \cdot FC : CH \cdot AF :: DB : AD$. (?)

$\therefore\ BH \cdot FC \cdot AD = CH \cdot AF \cdot DB$. (?) Q.E.D.

(This is Menelaus's Theorem, discovered about 80 B.C., and the following is its converse.)

984. If three points be given in the sides of a triangle, or the sides extended, so that the product of three non-adjacent segments is equal to the product of the other three, the three points are collinear.



Post. Let ABC be a triangle, and D, F, and H be three points so chosen that $BH \cdot FC \cdot AD = CH \cdot AF \cdot DB$.

To Prove. D, F, and H are collinear.

Cons. Draw HF and extend it to meet AB in some point as K.

Dem.

$$BH \cdot FC \cdot AK = CH \cdot AF \cdot KB. \quad (?)$$

$$BH \cdot FC \cdot AD = CH \cdot AF \cdot DB. \quad (?)$$

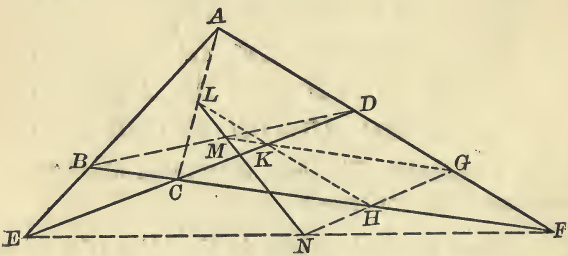
$$\therefore \frac{AK}{AD} = \frac{KB}{DB}. \quad (?)$$

$$\therefore AK \cdot DB = AD \cdot KB. \quad (?)$$

$\therefore$ points D and K must coincide, and

$\therefore$ D, F, and H are collinear. Q.E.D.

985. The mid-points of the three diagonals of a complete quadrilateral are collinear.



Post. Let $ABFDE$ be a complete quadrilateral, L, M, and N being the mid-points of the three diagonals AC, BD, and EF, joined by the lines LM and MN.

To Prove. L, M, and N are collinear.
Cons. Draw $NG \parallel$ to ED, join GM and LH.
Dem. GM is $\parallel$ to BF. (?)
$\therefore$ GM bisects CD. (?)
Similarly, LH is $\parallel$ to AF and bisects CD.
$\therefore$ LH, MG, and ED are concurrent in point K.
Regarding AE as a transversal of the triangle CDF,

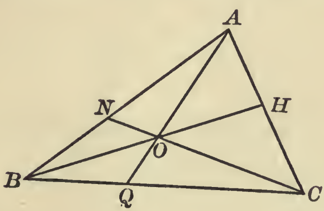
$$AD \cdot BF \cdot EC = AF \cdot BC \cdot ED. \quad (?)$$

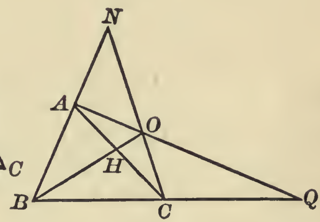
$$\therefore \frac{AD}{2} \cdot \frac{BF}{2} \cdot \frac{EC}{2} = \frac{AF}{2} \cdot \frac{BC}{2} \cdot \frac{ED}{2}. \quad (?)$$

$$\therefore LK \cdot MG \cdot NH = LH \cdot MK \cdot NG. \quad (?)$$

$\therefore$ L, M, and N are collinear. (?) Q.E.D.

986. If lines drawn from the vertices of a triangle are concurrent, the product of three non-adjacent segments of the sides equals the product of the other three.





Post. Let ABC be a triangle, O any point in its plane through which pass the lines AQ, CN, and BH.

To Prove. $BQ \cdot HC \cdot AN = QC \cdot AH \cdot NB.$

Dem. $BQ \cdot HC \cdot AO = BC \cdot AH \cdot QO.$

(BH is a transversal cutting $\triangle ACQ$.)

$QC \cdot AO \cdot NB = BC \cdot QO \cdot NA.$

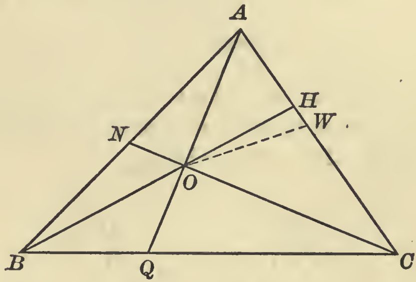
(NC is a transversal cutting $\triangle ABQ$.)

$$\therefore \frac{BQ \cdot HC}{QC \cdot NB} = \frac{AH}{NA}. \quad (?)$$

$$\therefore BQ \cdot HC \cdot NA = QC \cdot NB \cdot AH. \quad (?) \qquad \text{Q.E.D.}$$

(This is Ceva's theorem, discovered in 1678, and the following is its converse.)

987. If three points are given on the sides of a triangle, or the sides extended so that the product of three non-adjacent segments of the sides equals the product of the other three, the lines joining these points with the opposite vertices are concurrent.



Post. Let ABC be a triangle and N, Q, and H three points in the sides such that $BQ \cdot HC \cdot AN = QC \cdot AH \cdot NB$.

To Prove. AQ, BH, and CN are concurrent.

Cons. Extend BO to AC, meeting it in point W.

Dem. $$BQ \cdot WC \cdot AN = QC \cdot AW \cdot BN. \quad (?)$$

$$BQ \cdot HC \cdot AN = QC \cdot AH \cdot NB. \quad (?)$$

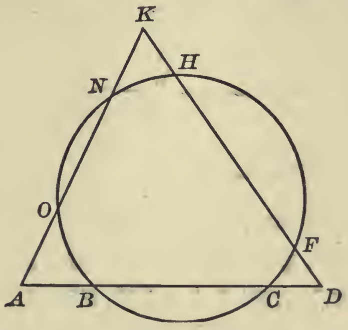
$$\therefore \frac{WC}{HC} = \frac{AW}{AH}. \quad (?)$$

$$\therefore WC \cdot AH = AW \cdot HC. \quad (?)$$

$\therefore$ points H and W coincide.

$\therefore$ AQ, BH, and CN are concurrent. Q.E.D.

988. If a circumference intersect the sides of a triangle, the product of the three non-adjacent secants and their external segments equals the product of the other three and their external segments.



Post. Let AKD be a triangle with its sides intersected by a circumference OHP.

To Prove. $(AB \cdot AC)(DF \cdot DH)(KN \cdot KO)$
$$= (DC \cdot DB)(KH \cdot KF)(AO \cdot AN).$$

Dem. $AB \cdot AC = AO \cdot AN.$ (?)

$DF \cdot DH = DC \cdot DB.$ (?)

$KN \cdot KO = KH \cdot KF.$ (?)

$\therefore (AB \cdot AC)(DF \cdot DH)(KN \cdot KO)$
$$= (DC \cdot DB)(KH \cdot KF)(AO \cdot AN). \quad (?)$$ Q.E.D.

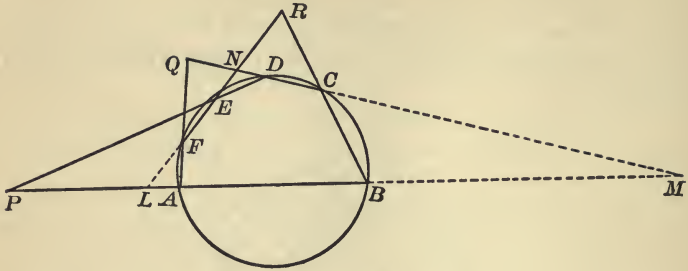
(This is known as Carnot's Theorem.)

989. If the opposite sides of an inscribed hexagon intersect each other the points of intersection are collinear.

Post. Let ABD be an inscribed hexagon with sides BA and ED meeting at P, CD and AF at Q, and BC and FE at R.

To Prove. P, Q, and R are collinear.

Cons. Extend RF to PB at L, and extend QC, and PB until they meet at M.



Dem. Regarding the triangle NLM.

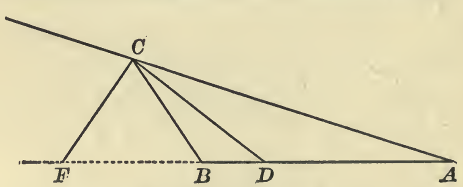
$$LP \cdot MD \cdot NE = PM \cdot DN \cdot EL, \qquad \text{(Menelaus)}$$
$$MQ \cdot NF \cdot LA = QN \cdot FL \cdot AM,$$
$$NR \cdot LB \cdot MC = RL \cdot BM \cdot CN.$$

$$\therefore (LP \cdot MQ \cdot NR)(MD \cdot MC)(NE \cdot NF)(LA \cdot LB) =$$
$$(PM \cdot QN \cdot RL)(DN \cdot CN)(EL \cdot FL)(AM \cdot BM)$$
$$(DM \cdot CM)(EN \cdot FN)(AL \cdot BL) =$$
$$(DN \cdot CN)(EL \cdot FL)(AM \cdot BM). \quad \text{(Carnot's Theorem.)}$$
$$\therefore LP \cdot MQ \cdot NR = PM \cdot QN \cdot RL. \quad (?)$$

$\therefore$ P, Q, and R are collinear. (?) Q.E.D.

(This theorem was discovered by Pascal in 1639, when he was 16 years old, and is therefore known as Pascal's Theorem.)

990. The bisectors of adjacent exterior and interior angles of a triangle divide the opposite side harmonically.



Dem. $BD : DA :: CB : CA$ (?)

$FB : FA :: CB : CA$ (?)

$$\therefore\ BD : DA :: FB : FA \quad (?)$$

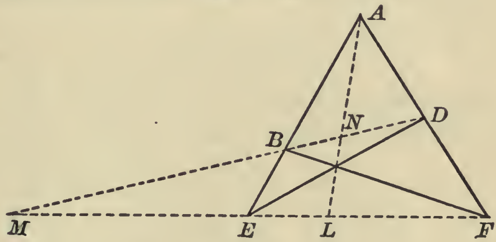
$\therefore$ BD is divided harmonically. Q.E.D.

Cor. CA and CB divide FD harmonically, for, taking the last proportion by alternation,

$$BD : FB :: DA : FA.$$

Consequently the four points, F, B, D, and A, are called *harmonic points*, and the two pairs, A and B, and F and D, are called *conjugate harmonic points*. Likewise the four lines, CF, CB, CD, and CA, are together called an *harmonic pencil*, and each line is called an *harmonic ray*.

991. In a complete quadrilateral, each diagonal is divided harmonically by the other two.



Dem. In $\triangle FAE$ the trans. DM gives

$$DF \cdot AB \cdot EM = AD \cdot BE \cdot MF, \quad (?)$$

$$DF \cdot AB \cdot EL = AD \cdot BE \cdot LF, \quad (?)$$

$$\therefore\ \frac{EM}{EL} = \frac{MF}{LF}. \quad (?)$$

$$\therefore\ EM : EL :: MF : LF.$$

$$\therefore\ EM : MF :: EL : LF.$$

$\therefore$ MF is divided harmonically.

In like manner it may be shown that AL and MD are divided harmonically. Q.E.D.

MAXIMA AND MINIMA.

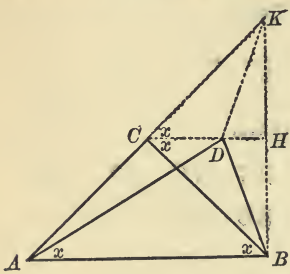
992. Among quantities of the same kind, that which is greatest is called the *maximum* (plural, *maxima*), and that which is the smallest is called the *minimum* (plural, *minima*). For example: of all lines inscribed in the same circle, the diameter is the greatest, and, therefore, is a *maximum;* and of all lines drawn from the same point to a given straight line, that which is perpendicular is the shortest, and is, therefore, a *minimum.*

If two or more plane figures have equal perimeters, they are said to be *isoperimetric.*

A plane figure is said to be a *maximum* or a *minimum* when its area is a *maximum* or a *minimum.*

THEOREMS.

993. If any number of triangles have the same or equal bases and equal areas, that which is isosceles has the *minimum perimeter.*



Post. Let ABC and ABD be two triangles having the same base AB and equal areas, and let ACB be isosceles, having CA equal to CB.

To Prove.

$$AC + CB + AB < AD + DB + AB,$$

or since $AB = AB,$

To Prove.

$$AC + CB < AD + DB.$$

Cons. From B construct a perpendicular to AB, and extend it to meet AC extended in K. Join DK and draw CH through point D.

Dem. How must the altitudes of the two $\triangle$ ACB and ADB compare?

What must be the position, then, of the line CH relative to AB?

How, then, do the two $\triangle HCB$ and CBA compare? Why?
The two $\triangle KCH$ and CAB? Why?
How, then, must the two $\triangle KCH$ and HCB compare? Why?
What is the position of CH relative to KB? Why?
How, then, must CK and CB compare? DK and DB?
Now compare $AC + CK$ with $AD + DK$, and the pupil should be able to write, or give orally, a complete and accurate demonstration of this theorem.

994. If any number of triangles have equal areas, that which is equilateral has the minimum perimeter.

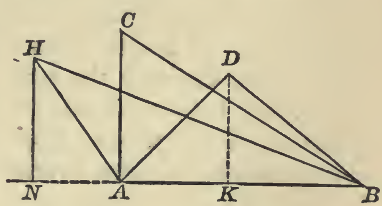
995. If any number of triangles have the same or equal bases and equal perimeters, that which is isosceles is the maximum.

Sug. Through the vertex of the isosceles triangle draw a line parallel to the base. Then prove that the vertex of the other triangle cannot fall on that line. Then compare their altitudes, and consequently their areas.

996. If any number of triangles be isoperimetric, that which is equilateral is the maximum.

997. If any number of triangles have two sides in each respectively equal, that in which these sides are perpendicular to each other is the maximum.

Sug. Place them so that one set of equal sides shall coincide, as AB. Then compare their altitudes DK, CA, and HN, and consequently their areas.



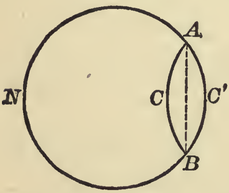
998. If any number of equivalent parallelograms have the same or equal bases, the perimeter of that which is rectangular is the minimum.

999. Of all rectangles of given area, the perimeter of a square is a minimum.

1000. If any number of triangles have the same or equal bases and the same or equal altitudes, the perimeter of that which is isosceles is a minimum.

1001. Every closed plane figure of given perimeter whose area is a maximum, must be *convex.*

Post. Let $ACBN$ be a plain concave figure, with the straight line AB which joins two of its points in its perimeter lying without.

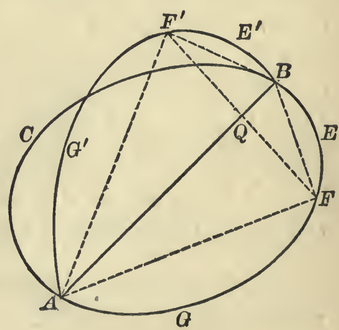


To Prove. That $ACBN$ cannot be a maximum.

Dem. Conceive the figure CAB to be revolved about AB as an axis till it comes to the position $AC'B$. Then the figures $ACBN$ and $AC'BN$ have equal perimeters, but the area of the latter will exceed that of the former. Hence $ACBN$ cannot be a maximum among isoperimetrical figures. But $ACBN$ is any concave, *i.e.* non-convex, plane figure. Therefore, of all isoperimetrical plane figures, the maximum must be convex.

1002. Of all plane figures that are isoperimetric, that which is a circle is the maximum.

Dem. I. It is evident that with a given perimeter, an indefinite number of figures of different shapes and areas may be constructed. It is also evident that we can diminish the area indefinitely, but cannot thus increase it. Consequently, there must be among all these figures having the same perimeter either one maximum figure, or several maximum figures of different forms and equal areas.



II. Let $ACBFG$ be a maximum figure with a given per-

imeter; then by 1001 it must be convex. Let also the line AB divide the perimeter into halves; then it must also divide the area into halves. For suppose one of the parts, as AFB, to be greater than the other, and conceive this part to be revolved on AB as an axis until it comes into the same plane with ACB, and let $AF'B$ be its position after revolution. Hence the perimeter of the figure $AF'BEGA$ is equal to that of the figure $ACBFGA$, but the area of the former is greater than that of the latter. Therefore the figure $ACBFG$ cannot be a maximum. But by hypothesis it is a maximum. Hence AB must bisect the area of $ACBFG$.

Since $ACBFG$ is a maximum, and AB bisects the area, it follows that the figure $AF'BFG$ is also a maximum.

Again, let F be any point in $BEGA$ selected at random, and F' its position after revolution. Join FF', FB, FA, $F'B$, and $F'A$. Then $AF = AF'$, and $FQ = F'Q$. Hence, the two triangles AQF and AQF' are equal.

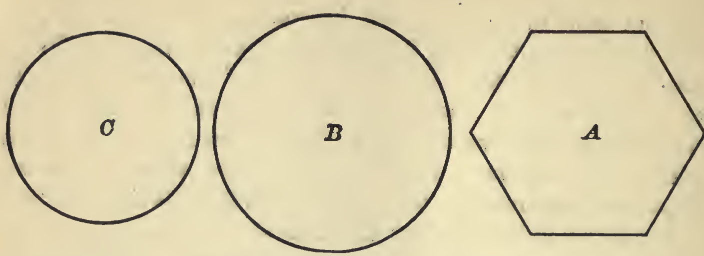
Therefore FF' is perpendicular to AB.

Similarly, the two triangles $AF'B$ and AFB are equal.

The triangle AFB must be a maximum, otherwise its area could be increased without increasing its perimeter; *i.e.* without increasing the lengths of the two chords AF and FB, which would consequently leave the areas of the two segments AGF and FEB unchanged, and therefore make up an area greater than $ABEG$, by which it is evident that $ACBFG$ could not be a maximum; but this also conflicts with the hypothesis which grants that $ACBFG$ is a maximum. Consequently the triangle AFB must be a maximum, and the angle AFB must be a right angle. (See Theorem 997.) But F is any point in the curve $BEFGA$. Hence BFA must be a semicircle, as also ACB. Hence the whole figure $ACBFG$ must be a circle. Q.E.D.

1003. **Of all plane figures having equal areas, the perimeter of that which is a circle is the minimum.**

Post. Let C be a circle, and A any other plane figure having the same area.



To Prove. Perimeter $C <$ perimeter A.

Dem. For let B be a circle having a perimeter equal to that of A. Hence by 1002 area B is greater than area A, and hence greater than area C. Hence if area C is less than area B, what must be true of their perimeters, *i.e.* their circumferences? But by construction

$$\text{Perimeter } B = \text{perimeter } A.$$

Hence perimeter of C is a minimum. Q.E.D.

1004. Of all mutually equilateral polygons, that which can be inscribed in a circle is the maximum.

Post. Let $ABCDH = P$, and $A'B'C'D'H' = P'$, be two mutually equilateral polygons, having AB equal to $A'B'$, BC equal to $B'C'$, etc., of which $ABCDH$ can be inscribed in a circle, and let N be the center of such circle.

Cons. With the radius NB construct the arcs $A'B'$, $B'C'$, $C'D'$, etc.

Dem. The arc $AB =$ arc $A'B'$, and arc $BC =$ arc $B'C'$, etc.

Hence circumference $ABCDH =$ sum of the arcs $A'B'$, $B'C'$, etc.

Hence perimeter of $S =$ perimeter of S'.

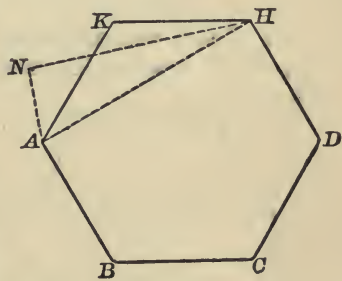
Therefore Area $S >$ area S'. (Theorem 1002.)

But the corresponding segments are equal. Hence, subtracting their respective sums from the above inequality leaves

Area $P >$ area P'. Q.E.D.

1005. Of all isoperimetric polygons of the same number of sides, that which is equilateral is a maximum.

Post. Let $ABCDHK$ be the maximum of all isoperimetrical polygons of any given number of sides.



To Prove. That it is equilateral; *i.e.* that

$KH = HD = DC$, etc.

Cons. Connect any two alternate vertices, as AH.

Dem. The $\triangle\ AKH$ must be a *maximum* of all isoperimetrical triangles having the common base AH; otherwise another triangle, as ANH, could be constructed, having the same perimeter and a greater area, in which case the area of the polygon $ABCDHN$ would be greater than that of $ABCDHK$. Hence the latter would not be a maximum. This result conflicts with our hypothesis, which grants that it is a maximum. Therefore the triangle AKH must be a maximum, and consequently *isosceles*. (See Theorem 995.)

Hence $AK = KH$.

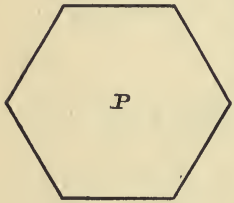
Similarly, by joining KD, $KH = HD$, etc.

Hence the polygon is equilateral. Q.E.D.

1006. **The maximum of all equilateral polygons of the same number of sides is that which is regular.**

1007. **Of all polygons having the same number of sides and equal areas, the perimeter of that one which is regular is a minimum.**

Post. Let P be a regular polygon, and M any irregular polygon having the same number of sides and same area as P; and let N be a regular polygon having the same number of sides and isoperimetrical with M.

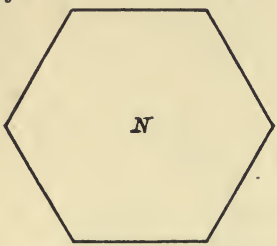


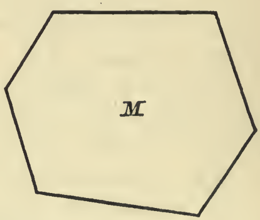
Dem. Area $M < N$. (See 1005 and 1006.)

Area $M =$ area P. Why?

$\therefore$ Area $P < N$.

But of two regular polygons of the same number of sides, that which has the less area must have the less perimeter? Why?





Hence Perimeter $P <$ perimeter N.

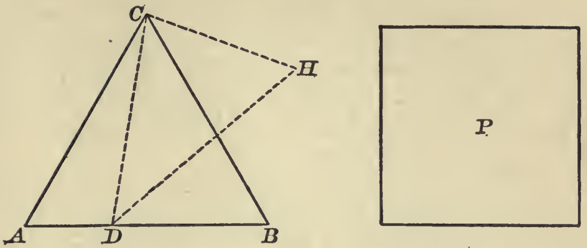
and $\therefore$ Perimeter $P <$ perimeter M.

Hence the perimeter of P is a minimum. Q.E.D.

1008. **Of all isoperimetric regular polygons, that which has the greatest number of sides is the maximum.**

Post. Let ABC be a regular *trigon*, and P a regular *tetragon*, having *equal* perimeters.

To Prove. Area $P >$ area ABC.



Cons. Draw from C any line CD to AB. At C make $\angle DCH$ equal to the $\angle CDB$, and CH equal to DB, and join HD.

Dem. $\triangle CDH = \triangle CDB$. Why?

Hence Area ABC = area $ADHC$. Why?

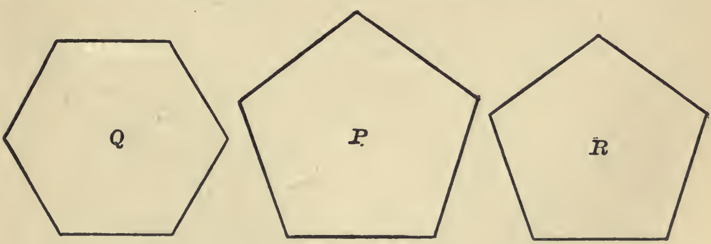
But Area $P >$ area $ADHC$. Why?

Hence Area $P >$ area ABC.

Similarly, P could be shown to be less than an isoperimetric regular pentagon, etc. Q.E.D.

1009. **The area of a circle is greater than the area of any polygon of equal perimeter.**

1010. **Of all regular polygons having a given area, the perimeter of that which has the greatest number of sides is a minimum.**



Post. Let Q and P be two regular polygons having equal areas, and Q having the greater number of sides.

To Prove. That the perimeter of P is greater than that of Q.

Dem. Let R be a regular polygon whose perimeter is equal to that of Q, but the number of sides the same as P.

Then $$Q > R. \quad \text{Why?}$$

But $$\text{Area } Q = \text{area } P.$$

$$\therefore \text{Area } P > \text{area } R.$$

$$\therefore \text{Perimeter } P > \text{perimeter } R. \quad \text{Why?}$$

But $$\text{Perimeter } R = \text{perimeter } Q.$$

$$\text{Perimeter } P > \text{perimeter } Q.$$

Hence perimeter Q is a minimum. Q.E.D.

1011. The circumference of a circle is less than the perimeter of any polygon of equal area.

1012. The rectangle formed by the two segments of a line is a maximum when the segments are equal.

SYMMETRY.

1013. Two points are said to be *symmetrical* with respect to a point when they are equidistant from, and in the same line with, this point. This point is called the *center of symmetry.*

1014. Two points are said to be *symmetrical* with respect to a line when the line that joins them is perpendicular to, and bisected by, this line. This line is called an *axis of symmetry.*

1015. Two points are said to be symmetrical with respect to a plane when the line that joins them is perpendicular to, and bisected by, this plane. This plane is called a *plane of symmetry.*

1016. The distance of either of two symmetrical points from the center of symmetry is called the *radius of symmetry.*

1017. Two plane figures are symmetrical with respect to a *center, axis,* or a *plane* when any point in either figure selected at random has a corresponding symmetrical point in the other.

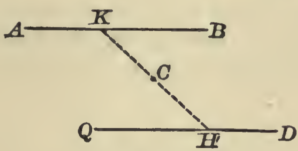


FIG. I.

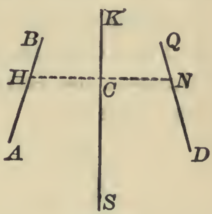


FIG. II.

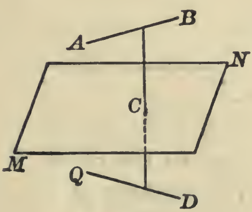


FIG. III.

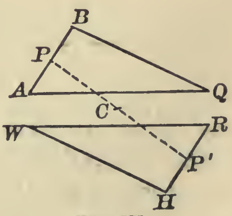


FIG. IV.

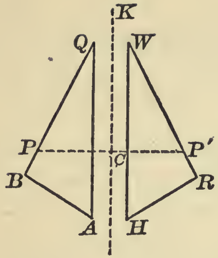


FIG. V.

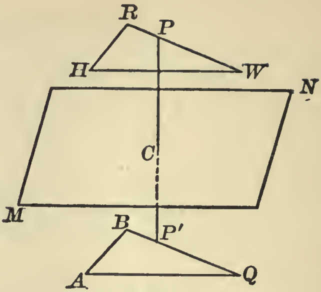


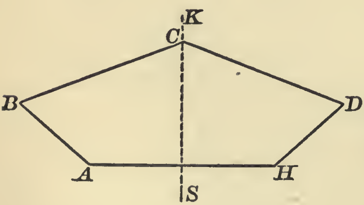
FIG. VI.

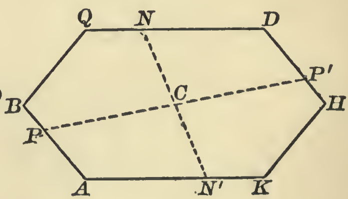
Thus, in Figs. I., II., and III. the lines *AB* and *QD* are symmetrical with respect to the center *C*, the axis *KS*, and the plane *MN*, respectively. In Figs. IV., V., and VI. the same is true of the triangles *ABQ* and *HRW*.

1018. A plane figure is symmetrical

i. when it can be divided by an axis into two figures symmetrical with respect to that axis.

ii. when it has a center such that, if a line be drawn through it in any direction at random, the two points at which it intersects the perimeter are symmetrical with respect to that center.





Thus figure *ABCDH* is symmetrical with respect to the axis *KS*, and *ABQDHK* with respect to the center *C*. In the latter case *PP′* or *NN′* is called a *diameter of symmetry*. (See 1016.)

THEOREMS.

1019. The center of a circle is a center of symmetry.

1020. The diameter of a circle is an axis of symmetry.

1021. The line which bisects the vertical angle of an isosceles triangle is an axis of symmetry.

1022. Either altitude of an equilateral triangle is an axis of symmetry.

1023. A segment of a circle is a symmetrical figure.

1024. That part of a circle included between two parallel chords is a symmetrical figure.

1025. The common part of two intersecting circles is a symmetrical figure.

1026. The diagonal of a square is an axis of symmetry.

1027. Every equilateral tetragon is a symmetrical figure.

(How many axes of symmetry does a square have?)

1028. The point of intersection of the diagonals of a parallelogram is a center of symmetry.

1029. An isosceles trapezoid is a symmetrical figure.

1030. If one diagonal of a tetragon divides it into two isosceles triangles, the other diagonal is an axis of symmetry.

1031. The bisector of an angle of a regular polygon is an axis of symmetry.

1032. The perpendicular bisector of one side of a regular polygon is an axis of symmetry.

1033. If the angles at the extremities of one side of an equilateral pentagon be equal, the pentagon is a symmetrical figure.

1034. If two diametrically opposite angles of an equilateral hexagon are equal, the hexagon is a symmetrical figure.

1035. Every equiangular tetragon is a symmetrical figure.

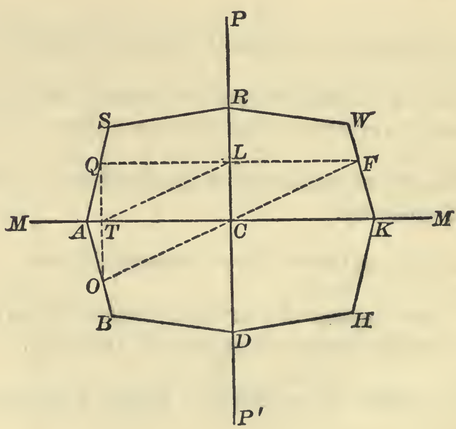
1036. If a figure have two axes of symmetry perpendicular to each other, their intersection is a center of symmetry.

Post. Let $ABDH$, etc., be a figure having two axes of symmetry PP' and MM' $\perp$ to each other.

To Prove. That their point of intersection C is a center of symmetry.

Cons. From any point in the perimeter selected at random, as Q, construct $QO \perp$ to MM', $QF \perp$ to PP', and join TL, OC, and FC.

Dem. $OT = TQ = LC$. Why?



Then what kind of a figure is $OTLC$? What relation, then, between TL and OC? Compare in a similar manner TL and CF. Finally compare OC and CF.

Hence points O and F are in the same line with and equidistant from C. Hence C is a center of symmetry. Q.E.D.

1037. METRIC AND COMMON MEASURES OF LENGTH AND SURFACE.

LINEAR MEASURE.

12	inches (in.)	= 1 foot (ft.).
3	feet	= 1 yard (yd.).
$5\frac{1}{2}$	yards, or $16\frac{1}{2}$ feet	= 1 rod (rd.).
320	rods, 1760 yards, or 5280 feet	= 1 mile (mi.).

MEASURES OF SURFACE.

144	square inches (sq. in.)	= 1 square foot (sq. ft.).
9	square feet	= 1 square yard (sq. yd.).
$30\frac{1}{4}$	square yards, or	= 1 square rod (sq. rd.).
$272\frac{1}{4}$	square feet	
160	square rods, or	= 1 acre (A.).
10	square chains	
640	acres	= 1 square mile (sq. mi.).

LINEAR MEASURE.

Divisions.	A **millimeter** (mm.)	= .001 of a meter.
	A **centimeter** (cm.)	= .01 of a meter.
	A decimeter	= .1 of a meter.
The unit.	A **meter** (m.).	
Multiples.	A dekameter	= 10 meters.
	A hektometer	= 100 meters.
	A **kilometer** (km.)	= 1000 meters.

MEASURES OF SURFACE.

A square millimeter (qmm.)	= .000001 of a square meter.
A square **centimeter** (qcm.)	= .0001 of a square meter.
A square decimeter	= .01 of a square meter.
A **square meter** (qm.)	= **principal unit.**
A square dekameter	= 100 square meters.
A square hektometer	= 10000 square meters.
A **square kilometer** (qkm.)	= 1000000 square meters.

In the measurement of land, the **square dekameter** is called an **are** (a.), and the **square hektometer** is called a **hektare** (ha.).

Tables of Equivalents.

Length.

Meter = 39.37043 in.	Inch = 2.53998 cm.
Meter = 1.09362 yd.	Yard = 0.91439 m.
Kilometer = 0.62138 mi.	Mile = 1.60933 km.

Surface.

Square Meter = 1550.031 sq. in.	Square inch = 6.45148 qcm.
Square Meter = 1.19601 sq. yd.	Square yard = 0.83611 qm.
Hektare = 2.47110 A.	Acre = 0.40468 ha.

Approximate Equivalents.

Meter = 1.1 yd.	Yard = .9 m.
Kilometer = $\frac{5}{8}$ mi.	Mile = 1.6 km.
Square meter = $1\frac{1}{5}$ sq. yd.	Square Yard = $\frac{5}{6}$ qm.
Hektare = $2\frac{1}{2}$ A.	Acre = $\frac{2}{5}$ ha.

ENTRANCE EXAMINATION PAPERS.

HARVARD — 1899.

New Method. — The University provides a Syllabus. The order of propositions belonging to one and the same Book is not prescribed; but it is not expected that a proposition of a given Book shall be proved by the aid of propositions appearing in later Books. Should the candidate, however, have used a text-book in which the division into Books is inconsistent with the division of the Syllabus, and should he prefer to follow the order of propositions with which he is familiar, he will be allowed to do so on stating in his examination book the name of the text-book he has used. Omit one of the starred questions.

*1. Prove that if two right triangles have the hypothenuse and a side of one respectively equal to the hypothenuse and a side of the other, the triangles are equal.

Two triangles have an angle, the opposite side, and another side of one equal respectively to an angle, the opposite side, and another side of the other. Are these triangles necessarily equal? Give roughly, by the aid of a figure, the reason for your answer without giving a formal proof.

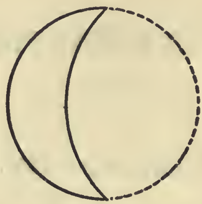
*2. Prove that two parallel straight lines intercept equal arcs on a circumference.

A straight line joins the centers of two circles and cuts the first circle in the points A and B, and the second circle in the points C and D. Four parallel lines pass through A, B, C, and D respectively. One half of the first circumference lies between the first two of these lines. What part of the second circumference lies between the second two lines, and what angle do these four lines make with the line AB?

*3. A set of circles are tangent to a given line at a given point; and a second set of circles are tangent to a second line at the same point. What is the locus of the points of intersection of equal circles of the two families? Prove that your answer is correct.

4. Three unequal circles are so situated that each of them is externally tangent to the other two. At the points of contact tangents are drawn. Prove that these three tangents meet in a point.

*5. A crescent-shaped region is bounded by a semicircumference of radius a, and another circular arc whose center lies on the semi-



circumference produced. Find the area and the perimeter of the region.

6. Prove that through any given straight line a plane can be passed perpendicular to any given plane.

Is it ever possible to pass more than one such plane through the line? If so, when? Can a plane always be passed through a given straight line perpendicular to a given line?

7. What is meant by the pole of a circle on a sphere?

Prove that all the points on the circumference of a circle on a sphere are equally distant from either of its poles.

Define the term "polar triangles."

8. AB, AC, AD are three edges of a cube which meet in the vertex A. A plane is passed through the middle points of these edges. If the cube contains 8 cubic feet, find the volume of the corner cut off by the plane, and the length of the perpendicular dropped from the center of the cube on the plane.

HARVARD — 1900.

1. Prove that in an isosceles triangle the angles opposite the equal sides are equal.

Given two lines AB and BC. Show how to draw through a point D of AB a third line making with BC the same angle which AB makes with BC.

2. Prove that an angle formed by a tangent and a chord is measured by one half the intercepted arc.

What is meant here by the statement that an angle is "measured by" half an arc?

A certain diameter of a circle makes with a tangent an angle of 70°. Find the angles which the tangent makes with the chords which join the point of contact with the extremities of the diameter.

3. Through a point A on the circumference of a circle, chords are drawn. On each one of these chords a point is taken one third of the distance from A to the other end of the chord. Find the locus of these points, and prove that your answer is correct.

4. Two circles intersect at the points A and B. Prove that they make equal angles with one another at A and at B.

[N. B. by the angle between two circles at their point of intersection is meant the angle between their tangents at this point.]

An area (the upper area in the figure) is bounded by three arcs of circles which, if produced, would meet in a point. Prove that the

sum of the angles between the circles at the vertices of this area is equal to two right angles.

5. Prove that the circumference of a circle is the limit which the perimeters of regular inscribed and circumscribed polygons approach when the number of their sides is increased indefinitely.

Find the perimeter of a regular hexagon inscribed in a circle of radius a; and also of a regular hexagon circumscribed about the same circle. What can you infer from these results concerning the value of π?

6. A square, each of whose sides is 5 inches long, has its corners cut off in such a way as to make it into a regular octagon. Find the area and the perimeter of the octagon.

HARVARD — 1901.

1. Prove that, if two angles of a triangle are equal, the triangle is isosceles.

The triangle ABC has a right angle at A. Through A a line is drawn, forming with AB an angle equal to B, and cutting BC in D. Prove that the two triangles ABD and ADC are isosceles, and that D is the middle point of BC.

*2. Prove that, when two tangents to the same circle intersect each other, the distances from their point of intersection to their points of contact are equal.

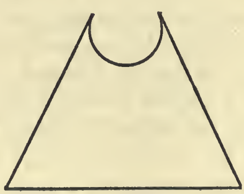
Two circles are tangent externally at A; BC is a common tangent, B and C being the points of contact. Prove that the angle BAC is a right angle.

Hint: Draw the common tangent at A.

3. Given a fixed point D within a triangle ABC. Choose any point, E, on the perimeter of the triangle, draw DE, and let P be the middle point of DE. Find the locus of P when E traces out the whole perimeter of the triangle. Describe the position of the locus exactly, and prove the correctness of your answer.

4. Two parallel tangents are drawn to a circle, and a third tangent is drawn intersecting these two in A and B, and tangent to the circle at C. Prove that the product of the segments AC and CB is equal to the square of the radius of the circle.

*5. Given a straight line and two points, one of which lies on the line. Show how to construct the center of a circle passing through the two points and tangent to the line at the given point of the line. Is there any case in which this construction is impossible?



*6. A piece is cut out of an equilateral triangle by means of an arc of a circle tangent to two sides. The side of the triangle is 7

inches and the radius of the circle 1 inch. Compute to two decimal places the perimeter and the area of the figure which is left.

7. Prove that, if two planes are perpendicular to each other, a straight line drawn in one of them perpendicular to their intersection is perpendicular to the other.

Is it true that planes perpendicular to the same straight line are parallel to each other? (Illustrate by a figure.)

Is it true that planes perpendicular to the same plane are parallel to each other? (Illustrate by a figure.)

8. Prove that the lateral area of a regular pyramid is equal to the product of the perimeter of its base and half its slant height.

Is this theorem true for an irregular pyramid? (Give the reason.)

9. A sphere of radius 13 inches has a cylindrical hole bored through it, the axis of the cylinder passing through the center of the sphere and the radius of the cylinder being 5 inches. Find the entire surface and the volume of the solid which is left.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY — 1900.

1. From a given point without a straight line, only one perpendicular can be drawn to the line.

2. If two triangles have two sides of one equal respectively to two sides of the other, but the third side of the first greater than the third side of the second, the included angle of the first is greater than the included angle of the second.

3. The medians of a triangle meet in a point.

4. A straight line can intersect a circumference in not more than two points.

5. In the same circle, or in equal circles, two central angles are in the same ratio as their intercepted arcs.

6. Two triangles are similar when their sides are parallel each to each, or perpendicular each to each.

7. If the triangle ABC has the side AB fixed, and the side AC twice the side BC, the vertex C lies on a circle which divides the line AB externally and internally in the same ratio.

8. Compute the difference in area between a circle of radius 3 and an inscribed regular hexagon.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY — 1901.

1. The perpendiculars from the vertices of a triangle to the opposite sides meet in a common point.

2. A circle is inscribed in a triangle ABC. If D, E, and F are the points of contact of the sides BC, CA, and AB, respectively, prove that the angle EDF is one half the supplement of the angle EAF.

3. Two circles whose diameters are 8 and 2 inches, respectively, touch each other externally. What is the length of their common tangent?

4. Define Similar Polygons. Show that the perimeters of two similar polygons are in the same ratio as any two homologous sides.

5. In equal circles or in the same circle the greater of two chords subtends the greater arc.

6. Show how to construct a triangle equivalent to a given polygon.

7. If two triangles have two sides of the one equal, respectively, to two sides of the other, but the third side of the first greater than the third side of the second, the included angle of the first is greater than the included angle of the second.

8. If the radius of a circle is 6, what is the area of a segment whose arc is 60°?

BROWN UNIVERSITY—1900.

1. In the same or in equal circles the greater of two unequal chords is nearer the center.

2. The areas of two rectangles having equal altitudes are to each other as their bases.

3. A line drawn from a vertex of a triangle to the middle point of the opposite side is less than half the sum of the adjacent sides.

4. Two regular polygons of the same number of sides are similar.

5. Find the side of a square inscribed in a circle whose area is 98π.

BROWN UNIVERSITY — 1901.

1. Through any point not in a line a perpendicular can be drawn to the line.

2. The bisectors of the angles of a triangle meet in a point.

3. Two triangles having an angle of one equal to an angle of the other are to each other as the products of the sides including the equal angles.

4. A regular triangle is circumscribed about a circle whose radius is 7. Find the area of that portion of the triangle which is outside of the circle.

YALE — 1899.

1. Two triangles are similar if their homologous sides are proportional.

2. Define the limit of a variable. Prove that if two variables are always equal their limits are equal. Prove that the area of a circle is equal to one half the product of its radius and circumference.

3. (*a*) In any quadrilateral if a line be drawn through the middle points of two adjacent sides, and a second line through the middle points of the other two sides, these lines will be parallel.

(*b*) If the middle points of the opposite sides of a quadrilateral be joined, the lines so drawn will bisect each other.

4. (*a*) If two circles are tangent internally and if the radius of the one be the diameter of the other, a chord of the larger drawn through the point of tangency is bisected by the smaller.

(*b*) What example of a locus is found in the previous figure?

YALE — 1900.

[Time, One hour.]

In Question 1 *the work will not be accepted unless the constructions are made accurately with ruler and compass.*

1. Divide a line externally in extreme and mean ratio and prove the construction.

What use is made of this construction in Geometry?

2. (*a*) From a point A on the circumference of a circle two equal chords, AB and AC, are drawn. Prove that they make equal angles with the diameter through A.

(*b*) If each of these angles is 30°, prove that the points A, B, and C trisect the circumference.

3. In two similar triangles the bases and altitudes are proportional, and in two equivalent triangles the bases and altitudes are inversely proportional.

4. (*a*) Prove that the ratio of the circumference of a circle to its diameter is the same for all circles. What is the approximate value of this ratio?

(*b*) Explain briefly (without proofs) the method of determining this value.

YALE — 1901.

1. State and prove a theorem relating to the squares of the sides of an obtuse-angled triangle.

2. Construct accurately with ruler and compass two tangents to a circle which shall include between their points of contact an arc of 120° of the circumference, and write the value of the angle included between them.

The proof may be omitted if the method is made clear by the construction of the figure.

3. Define locus of a point.

Prove that the bisector of an angle A of a triangle and the bisectors of its exterior angles B and C meet in a point.

4. An isosceles triangle with vertex A is inscribed in a circle, and through A a line is drawn cutting the side BC at E and the circle at D. Prove that AB is a mean proportional between AD and AE.

5. Quote a theorem, a definition, and an axiom used in proving that the area of a triangle is equal to one half the product of its base and altitude. *Write nothing else.*

SHEFFIELD SCIENTIFIC SCHOOL — 1899.

[Note. — State at the head of your paper what text-book you have studied on the subject and to what extent.]

1. (*a*) The sum of the angles of a triangle equals two right angles.

(*b*) State and prove the theorem on the sum of the angles of any polygon.

2. If two circumferences intersect, the straight line joining their centers bisects their common chord at right angles.

State the corresponding theorem when the circumferences are *tangent* to each other.

3. A straight line parallel to one side of a triangle divides the other two sides proportionally.

4. When is a straight line divided in *extreme and mean ratio?*

Give and prove the construction for dividing a given straight line in extreme and mean ratio.

To the construction of what regular polygon does this construction apply?

5. What is the meaning of π in geometry?

Find the length of the side of an equilateral triangle whose area equals that of a circle of radius R.

SHEFFIELD SCIENTIFIC SCHOOL — 1900.

[Note. — State at the head of your paper what text-book you have studied on the subject and to what extent.]

1. The three perpendiculars erected at the middle points of the sides of a triangle meet in a common point.

2. State and prove the theorems regarding the measurement of the angle between (*a*) two chords of a circle; (*b*) two secants of a circle.

3. When are two magnitudes *commensurable?* when *incommensurable?*

Prove that the areas of two rectangles with equal bases are in the same ratio as the altitudes, both when the latter are commensurable and incommensurable.

4. The areas of two triangles having an angle of the one equal to an angle of the other are to each other as the products of the sides including the equal angles.

5. Given a square the length of whose side is 6 units, construct a rectangle with altitude 2 units and equivalent to the square.

6. What is the meaning of π in geometry? State and prove the theorem on the area of a circle.

UNIVERSITY OF PENNSYLVANIA — 1901.

[Time, Two hours.]

1. Define: A curve, parallel straight lines, polygon inscribed in a circle, similar figures, harmonic division of a straight line.

2. From a point without a straight line, one perpendicular can be drawn to that line, and but one.

3. In a trapezoid the straight line joining the middle points of the non-parallel sides is parallel to the bases, and is equal to one half their sum.

4. An angle formed by two secants intersecting without the circumference is measured by one half the difference of the intercepted arcs.

5. In any triangle, the sum of the squares of the two sides is equal to twice the square of half the base increased by twice the square of the medial line.

6. Triangles on the same or equal bases and between the same parallels are equivalent.

7. A circle may be circumscribed about any regular polygon; and a circle may also be inscribed in it.

8. Problem: To find two straight lines in the ratio of the areas of two given polygons.

DARTMOUTH — 1900.

1. Prove that the three perpendicular bisectors of the sides of any triangle meet in a common point.

2. Circumscribe a circle about a given triangle.

3. Construct a circle having its center in a given line, and passing through two given points.

4. Prove, algebraically, that the square of the side opposite the obtuse angle of an obtuse-angled triangle is equal to the sum of the squares of the other sides plus the product of one of these sides and the projection of the other side upon it.

5. If the side of an equilateral triangle is a, find its area.

6. Prove that the area of a circle is equal to half the product of its radius and circumference.

7. The radius of a circle is 2; find the area of the regular inscribed dodecagon.

8. The radius of a circle is R; what is the radius of a concentric circle which divides it into two equivalent parts?

DARTMOUTH — 1901.

1. Prove that the diagonals of a parallelogram bisect each other. When are they equal?

2. How many degrees in one angle of a regular decagon? of a regular dodecagon? What is the largest number of degrees possible in one angle of a regular polygon?

3. One angle between two chords intersecting within a circle is 50°; its intercepted arc is 10°. How many degrees in the arc intercepted by its vertical angle?

4. Prove that the bisector of an angle of a triangle divides the opposite side into segments proportional to the sides of the angle.

5. The diagonals of a rhombus are 9 and 12. Find its perimeter and area.

6. In a circle whose radius is R, show that the area of the inscribed square is $2\,R^2$, and of the inscribed regular dodecagon is $3\,R^2$.

7. Draw the tangents to a circle from a point outside the circle, and prove that they are equal.

PRINCETON — 1901.

Geometry (a).

1. Two triangles are equal when a side and two adjacent angles of the one are equal respectively to a side and two adjacent angles of the other.

2. The sum of the interior angles of a polygon is equal to two right angles, taken as many times less two as the figure has sides.

3. The tangents to a circle from an exterior point are equal.

4. If any chord is drawn through a fixed point within a circle, the product of its segments is a constant for all positions of the chord.

5. Through any point P two lines are drawn; on one of these, two points D and R are taken on opposite sides of P, and on the other line any point C is chosen; then a circle is described through the points C, D, and P, and also another circle through the point R and tangent to the first circle at the point P; the line CP is produced to meet the second circle at the point K. Prove the angle RKP is equal to the angle PCD.

Geometry (*b*). Mensuration.

Use logarithms in each exercise.

1. In a right triangle one side is 5.06 meters and the hypothenuse is 12.4 meters; what is the area of the triangle?

2. The areas of two similar triangles are 3.42 square meters and 8.04 square meters, and the altitude of the smaller is 1.25 meters; what is the altitude of the other triangle?

3. A quadrilateral of perimeter 36.4 meters is circumscribed about a circle of radius 3.62 meters; what is the area of the portion of the quadrilateral without the circle?

4. Find the area of the ring enclosed between two concentric circles of radii 4.32 meters and 3.41 meters.

PRINCETON SCHOOL OF SCIENCE — 1901.

1. Prove that the perpendicular bisectors of the sides of a triangle meet in a point, and that this point is equidistant from the vertices of the triangle.

2. Define when a plane figure is symmetrical with respect to a line; also with respect to a point.

Prove that if a figure is symmetrical with respect to two lines perpendicular to each other, it is also symmetrical with respect to their intersection.

3. Show how to construct geometrically a fourth proportional to three given straight lines; also how to construct a triangle when two sides and the included angle are given.

4. In any plane triangle state and prove what the square of the side opposite an acute angle is equal to.

5. Prove what the area of a triangle is equal to; also the area of a trapezoid. Define a trapezoid.

6. Show and prove how to construct a square equivalent to a given parallelogram; also a square equivalent to a given triangle.

7. Define a regular polygon. Prove that the perimeters of two regular polygons of the same number of sides are to each other as the radii of their circumscribed circles; and that their areas are as the squares of these radii.

WORCESTER POLYTECHNIC INSTITUTE — 1901.

1. Demonstrate: If from a fixed point without a circle a secant be drawn, the product of the secant and its external segment is constant, in whatever direction the secant is drawn.

2. Divide a given straight line into parts proportional to any number of given lines. Give proof.

3. (*a*) When is a line said to be divided into extreme and mean ratio? (*b*) What is a rhombus? (*c*) A regular decagon? (*d*) The median of a triangle? (*e*) An oblique angle?

4. Find the angle at the center of a circle subtended by an arc 6 feet 5 inches long, if the radius of the circle is 8 feet 2 inches.

5. Demonstrate: If two circles intersect and a secant is drawn through each point of intersection, the chords which join the extremities of the secants are parallel.

[NOTE. — No credit will be given for statements made in the course of a demonstration, unless the proper reasons therefor are also given.]

CORNELL — 1901.

1. Define: A point, a straight line, a plane, two parallel straight lines, a plane angle, a rectangle, a regular polygon.

2. If two sides of a triangle be unequal, the opposite angles are unequal, and the greater angle lies opposite the longer side; and conversely.

3. At a given point to construct an angle equal to a given angle.

4. To inscribe a circle in a given triangle, and to escribe three circles to it.

5. Three or more parallels will cut any two straight lines, not parallel to them, in proportional segments.

6. To construct a polygon that shall be similar to one given polygon and equal in area to another.

7. Two regular polygons of the same number of sides are similar.

8. If the radius of a circle be 8 feet, find the area of a segment cut off by the side of an inscribed equilateral triangle.

AMHERST — 1901.

1. The line joining the middle points of the non-parallel sides of a trapezoid is parallel to the bases, and equal to one half their sum.

2. The angle between two secants, intersecting without the circumference, is measured by one half the difference of the intercepted arcs.

3. Two polygons are similar when they are composed of the same number of triangles, similar each to each, and similarly placed.

4. To construct a square equivalent to twice a given square.

5. The perimeters of two regular polygons of the same number of sides are to each other as their radii, or as their apothems.

6. The area of an equilateral triangle is $9\sqrt{3}$. Find its side.

WELLESLEY COLLEGE — 1901.

1. State three cases in which triangles are similar. Prove one of them.

2. A circle can be circumscribed about any regular polygon.

3. If one of the equal sides of an isosceles triangle be the diameter of a circle, the circumference will bisect the base.

4. Solve one:

(*a*) The non-parallel sides of a trapezoid are 13 each; the parallel sides are 50 and 26. Find its area.

(*b*) Find the area of an equilateral triangle whose side is 12.

5. If two triangles have two sides of the one equal respectively to two sides of the other but the included angles unequal, what conclusion can be drawn? Prove.

6. Construct a mean proportional to two given straight lines, giving reasons for work.

7. If two chords intersect within a circle, what is the measure of the angle formed? Prove.

SMITH COLLEGE — 1901.

1. The sum of two lines drawn from a point to the extremities of a straight line is greater than the sum of two other lines similarly drawn but included by them.

2. Through a given point without a circle construct a tangent.

3. (Original.) The bisectors of the angles of a parallelogram form a rectangle.

4. Given a pentagon, construct an equivalent triangle.

5. Given a radius, and a side of a regular inscribed polygon, find the side of a regular polygon of double the number of sides.

6. (Original.) The sum of the perpendiculars from any point within a convex equilateral polygon upon the sides is constant.

VASSAR COLLEGE — 1901.

1. If the angle at the vertex of an isosceles triangle is 50°, find the exterior angle formed by producing the base.

2. Find the value of each angle of an equiangular decagon.

3. Two tangents drawn to a circle from an exterior point form an angle of 60°. How many degrees are there in the arc contained between their points of contact?

4. Prove that the opposite angles of an inscribed quadrilateral are supplementary.

5. Prove that triangles which have an angle in each equal are to each other as the rectangles of the including sides.

6. Define and illustrate by figures supplementary and complementary angles, similar polygons. Give all the conditions of the equality of the triangles.

7. How many tiles 9 inches long and 4 inches wide will be required to pave a path 8 feet wide surrounding a rectangular court 60 feet long and 36 feet wide?

8. Two tangents drawn from a point exterior to a circle whose radius is 1.3 inches intercept upon the circumference an arc whose length is 0.57 inches. Required the angle formed by the two tangents.

UNIVERSITY OF CHICAGO—1901.

[Time Allowed, One Hour and Thirty Minutes.]

[When required, give all reasons in full, and work out proofs and problems in detail.]

1. The square on the side of a triangle which lies opposite a right angle is equivalent to the sum of the squares of the other two sides.

(1) Give a proof of this theorem *different* from the one given in your text-book.

(2) State how this theorem should be modified when the given side lies opposite (a) an acute angle, (b) an obtuse angle.

(3) If unable to comply with the condition (1), give a proof for either (a) or (b) in (2).

2. Prove that the area bounded by the circumferences of two concentric circles of unequal radii is equivalent to the area of a circle whose diameter is that chord of the outer circumference which is tangent to the inner circumference.

3. Two tangents to a circle meet at an angle of 60°. If the radius of the circle is 21 inches, find the length of the arc included between the points of tangency of the two tangents. [Use $\pi = 3\frac{1}{7}$.]

4. Prove that one of the angles formed at the intersection of the two bisectors of the base angles of an isosceles triangle is equal to one of the exterior base angles of the triangle.

MICHIGAN SCHOOL OF MINES—1901.

1. Two triangles are equal when ——. State and prove.

2. The sum of the interior angles of a polygon is equal to ——. State and prove.

3. In the same or equal circles two angles at the center have the same ratio as the arcs which they subtend (two cases). Prove.

4. The square on the hypothenuse of a right-angled triangle is equivalent to the sum of the squares on the other two sides. Prove.

5. A line drawn from the vertex of a triangle bisecting the vertical angle divides the base into segments which are proportional to the adjacent sides. Prove.

6. The area of a circle is equal to ——. State and prove.

WASHINGTON UNIVERSITY — 1900.

1. Prove that two angles whose sides are respectively perpendicular to each other are either equal or supplementary.

2. Prove that an exterior angle of a triangle is equal to the sum of the two opposite interior angles.

3. Prove that the three lines drawn from the vertices of the angles of a triangle to the middle points of the opposite sides meet at a point.

4. If we have the proportion

$$\frac{a}{b}=\frac{m}{n}, \text{ prove that we also have } \frac{a+b}{a-b}=\frac{m+n}{m-n}.$$

Express this in words, as a theorem.

5. Prove that the line which bisects an angle of a triangle divides the opposite side into segments proportional to the adjacent sides.

6. Suppose two intersecting straight lines cut across the same circle. Prove that the angle between them is measured by one half the difference of the two intercepted arcs. [Draw the figure so that one of the lines passes through the center of the circle, while the other intersects it without the circle.]

7. Prove that if two triangles have their homologous sides proportional, their homologous angles are equal.

UNIVERSITY OF CALIFORNIA — 1901.

1. Prove that the sum of the angles of a triangle is equal to two right angles. State a very important axiom (postulate) upon which the proof of this proposition depends.

2. Prove that the angle contained by the bisectors of two exterior angles of any triangle is equal to half the sum of the two corresponding interior angles.

3. Construct a triangle having given its base, one of its sides, and its altitude.

4. Prove that the areas of similar triangles are to one another as the squares described on their homologous sides. Show that this proposition is also true of similar polygons.

5. Two chords intersect within a circle; prove that the rectangle contained by the segments of one of the chords is equal to the rectangle contained by the segments of the other.

6. From a point P, outside a circle, a straight line is drawn cutting the circumference in M and N so that $PN = kPM$. The radius of the circle is r and the distance of P from the center is d. Find PM in terms of k, r, and d.

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